Ph.D. Qualifying Examination
Sem 2, 2002/2003
Linear Algebra

(1) Let V be a real vector space and we define

$$V^* = \{f : V \to \mathbb{R} : f \text{ is } \mathbb{R}\text{-linear}\}.$$

We define vector addition and scalar multiplication on V^* by

$$\begin{aligned} (f_1 + f_2)(v) &= f_1(v) + f_2(v) \\ (af_1)(v) &= a(f_1(v)) \end{aligned}$$

for all $v \in V$, $a \in \mathbb{R}$ and $f_1, f_2 \in V^*$. You may assume that these define a vector space on V^*.

Definition. The vector space V^* is call the *dual space* of V.

Similarly using V^*, we define the vector space $(V^*)^*$.

We define a function $\phi : V \to (V^*)^*$ by $(\phi(v))(f) = f(v)$ for $v \in V$ and $f \in V^*$.

(a) Suppose V is finite dimensional, show that $\dim V = \dim V^*$.
(b) Suppose V is finite dimensional, show that ϕ is a bijection.
(c) Give an example where V is an infinite dimensional vector space and ϕ is NOT a bijection.

[25 marks]

(2) In this question, we will develop the concept of tensor products.

Let V and W be real vector spaces of dimension m and n respectively.

Definition. Let X be a real vector space. A function $f : V \times W \to X$ is called a *bilinear* function if

$$\begin{aligned} f(a\mathbf{v} + a'\mathbf{v}', \mathbf{w}) &= af(\mathbf{v}, \mathbf{w}) + a'f(\mathbf{v}', \mathbf{w}) \\ f(v, b\mathbf{w} + b'\mathbf{w}') &= bf(\mathbf{v}, \mathbf{w}) + b'f(\mathbf{v}, \mathbf{w}') \end{aligned}$$

for all $a, a', b, b' \in \mathbb{R}$, $\mathbf{v}, \mathbf{v}' \in V$ and $\mathbf{w}, \mathbf{w}' \in W$.

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a basis of V and let $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be a basis of W.

Let U be a vector space with basis

$$\{\mathbf{u}_{ij} : i = 1, \ldots, m \text{ and } j = 1, \ldots, n\}.$$

We define a bilinear function $\Phi : V \times W \to U$ such that

$$\Phi(\mathbf{v}_i, \mathbf{w}_j) = \mathbf{u}_{ij}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Definition The real vector space U is called the *tensor product* of V and W and we denote it by $V \otimes_{\mathbb{R}} W$.

Universal property of tensor products.

- Let X be a real vector space and let $f : V \times W \to X$ be a bilinear function. Then there exists a **unique** linear transformation $F : U \to X$ such that $f = F \circ \Phi$.

(a) Prove the universal property of tensor products.

(b) Suppose U' is a real vector space and suppose $\Phi' : V \times W \to U'$ is a bilinear function. Suppose U' satisfy the universal property too, namely

- For every vector space X and every bilinear function $f : V \times W \to X$, there exists a unique linear transformation $F' : U' \to X$ such that $f = F \circ \Phi'$.

Show that there is a *bijective* linear transformation $L' : U \to U'$.

[25 marks]

(3) Let V be a two dimensional complex vector space. Define

$$\mathrm{GL}(V) := \{f : V \to V | f \text{ is } \mathbb{C}\text{-linear and invertible}\}.$$

Let P_n denote the $\mathbb{C}$-linear space of all degree n complex polynomials on V. Let $g \in \mathrm{GL}(V)$ and $f \in P_n$. We define $f_g \in P_n$ to be the polynomial such that

$$f_g(v) = f(g^{-1}(v))$$

for all $v \in V$. Suppose W is a **nonzero** complex vector subspace of P_n such that for every $f \in W$, then $f_g \in W$ for every $g \in \mathrm{GL}(V)$. Show that $W = P_n$.

Remark. The space P_n is known as the symmetric n-th power of the dual representation to the standard representation of $\mathrm{GL}(V)$. [25 marks]

(4) Let V be a complex vector space of dimension n. Let $\sigma : V \to V$ denote a $\mathbb{R}$-linear transformation. We say that σ is a *complex conjugation* on V if

- σ^2 is the identity linear transformation on V

- $\sigma(z\mathbf{v}) = \overline{z}\sigma(\mathbf{v})$ for all $z \in \mathbb{C}$ and $\mathbf{v} \in V$.

(a) Show that
$$V^\sigma := \{\mathbf{v} \in V : \sigma(\mathbf{v}) = \mathbf{v}\}$$
is a $\mathbb{R}$-vector subspace of V of dimension n.

(b) Let $\langle \mathbf{v}, \mathbf{v}' \rangle$ denote a non-degenerate symmetric $\mathbb{C}$-bilinear form on V, that is,
- $\langle \mathbf{v}, \mathbf{v}' \rangle = \langle \mathbf{v}', \mathbf{v} \rangle \in \mathbb{C}$ for all $\mathbf{v}, \mathbf{v}' \in V$.
- The form $\langle \mathbf{v}, \mathbf{v}' \rangle$ is $\mathbb{C}$-linear in $\mathbf{v}$ and $\mathbf{v}'$.
- Suppose $\mathbf{v} \in V$ satisfies $\langle \mathbf{v}, \mathbf{v}' \rangle = 0$ for all $\mathbf{v}' \in V$, then $\mathbf{v} = \mathbf{0}$.

Let p, q be nonnegative integers such that $p + q = n$. Show that there exists a complex conjugation σ on V such that
- $\overline{\langle \mathbf{v}, \mathbf{v}' \rangle} = \langle \sigma(\mathbf{v}), \sigma(\mathbf{v}') \rangle$ for all $\mathbf{v}, \mathbf{v}' \in V$ and
- the restriction of $\langle \mathbf{v}, \mathbf{v}' \rangle$ to V^σ is a symmetric $\mathbb{R}$-bilinear form with signature (p, q).

We recall that a symmetric form with signature (p, q) means that there exists a $\mathbb{R}$-basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of V^σ such that $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ if $i \neq j$, $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = 1$ if $i \leq p$ and $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = -1$ if $i \geq p + 1$.

(c) Suppose $\dim_{\mathbb{C}} V \geq 2$ and let σ' be another complex conjugation on V satisfying the conditions in (b). Is $V^\sigma = V^{\sigma'}$? If it is true, give a proof. If it is false, give a counterexample.

[25 marks]

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