**Ph.D. Qualifying Examination**
**Sem 2, 2002/2003**
**Complex Analysis**

1. Does there exist a domain $D$ in $\mathbb{C}$ and a function $f$ such that $f$ is analytic on $D$ and
$$\mathrm{Re}[f(x+iy)] = e^{x+y}$$
for all $x+iy \in D$? Justify your answer. [10 marks]

2. Prove that the equation
$$e^{-z} + z = 2$$
has exactly one root in the right half-plane, and this root is real. [18 marks]

3. Let $D$ be an open subset of $\mathbb{C}$ and $z_0 \in D$. Prove that if $f$ is analytic on $D$ and $f'(z_0) \neq 0$, then
$$\frac{2\pi i}{f'(z_0)} = \int_\gamma \frac{1}{f(z) - f(z_0)}\, dz,$$
where $\gamma$ is a small circle centred at $z_0$. [18 marks]

4. Find a function $g$ with the properties:
   (i) $g$ is analytic on $D = \{z \in \mathbb{C} : \ |z| > 2\}$.
   (ii) $[g(z)]^2 = 4z^2 - 9$ for all $z \in D$.

[18 marks]

5. Given that
$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$
Let $\lambda > 0$. By considering the integral
$$\int_\gamma e^{-z^2}\, dz$$
where $\gamma$ is the boundary of the rectangle with vertices at $0$, $R$, $R+\lambda i$ and $\lambda i$, prove that

(i) $\int_0^\infty e^{-x^2} \cos(2\lambda x)\ dx = \frac{\sqrt{\pi}}{2} e^{-\lambda^2}.$

(ii) $\int_0^\infty e^{-x^2} \sin(2\lambda x)\ dx = e^{-\lambda^2} \int_0^\lambda e^{y^2}\ dy.$

[18 marks]

6. Prove that if the function $f$ is analytic in a deleted neighborhood $D$ of $z = 0$ and $f(1/n) = 0$ for all nonzero integers $n$, then either $f$ is identically zero in $D$ or $f$ has an essential singularity at $z = 0$. [18 marks]

**END OF PAPER**