Ph.D. Qualifying Examination
Sem 2, 2002/2003
Analysis

(1) [20 marks]

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f'(-1) < 2$ and $f'(1) > 2$, show that there exists $x_0 \in (-1, 1)$ such that $f'(x_0) = 2$. (Hint: consider the function $f(x) - 2x$ and recall the proof of Rolle's theorem)

(b) Let $f : (-1, 1) \to \mathbb{R}$ be a differentiable function on $(-1, 0) \cup (0, 1)$ such that $\lim_{x \to 0} f'(x) = l$. If f is continuous on $(-1, 1)$, show that f is indeed differentiable at 0 and $f'(0) = l$.

(2) Let $\mathbb{P}_n$ be the space of polynomials of degree $\leq n$ on $\mathbb{R}$ for each $n \in \mathbb{N}$. If $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, define [20 marks]

$$||p||_M = \max\{|a_0|, |a_1|, \cdots, |a_n|\}$$

$$||p||_\infty = \max\{|p(x)| : x \in [0, 1]\}, \text{ and } ||p||_1 = \int_0^1 |p(x)| dx.$$

(i) Show that $|| \cdot ||_1$ is a norm of the space $\mathbb{P}_n$.

(ii) Use the fact that $|| \cdot ||_M$ and $|| \cdot ||_\infty$ are also norms of $\mathbb{P}_n$, or otherwise, to show that there exists a positive constant c_n such that

$$c_n ||p||_\infty \leq ||p||_1 \leq (1/c_n) ||p||_M$$

for all $p \in \mathbb{P}_n$. (Hint: note that $\mathbb{P}_n \equiv \mathbb{R}^{n+1}$)

(iii) With the help of the Weiestrass approximation theorem, show that there is no positive constant c such that $c_n > c$ for all n. (Note that for each $\varepsilon > 0$, there is a nonnegative continuous function f_ε on $[0, 1]$ such that $f_\varepsilon(0) = 1$ and $||f_\varepsilon||_1 < \varepsilon$.)

(3) Prove or disprove each of the following statements. [40 marks]

(a) If $f : [1,5] \to [1,5]$ is a continuous function, then there exists $x_0 \in [1,5]$ such that $f(x_0) = x_0$.

(b) Let $\{f_n\}$ be a sequence of uniformly continuous functions on an interval I. If $\{f_n\}$ converges uniformly to a function f on I, then f is also uniformly continuous on I.

(c) Let $\{f_n\}$ be a sequence of functions that converges uniformly to a function f on (0,2). If each of the f_n is differentiable on (0,2), then f is also differentiable on (0,2).

(d) If f is a continuous function on [-1,1], then there exists a constant $M > 0$ such that

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2| \text{ for all } x_1, x_2 \in [-1,1].$$

(e) If f is a uniformly continuous function on $(0,5)$, then there exists a positive number ε such that the function $g(x) = 1/(f(x) + \varepsilon)$ is also uniformly continuous on $(0,5)$.

(4) Let $g : [0,1] \times [0,1] \to [0,1]$ be a continuous function and let $\{f_n\}$ be a sequence of functions such that [20 marks]

$$f_n(x) = \begin{cases} 0, & 0 \leq x \leq 1/n, \\ \displaystyle\int_0^{x-\frac{1}{n}} g(t, f_n(t))dt, & 1/n \leq x \leq 1. \end{cases}$$

With the help of the Arzela-Ascoli theorem or otherwise, show that there exists a continuous function $f : [0,1] \to \mathbb{R}$ such that

$$f(x) = \int_0^x g(t, f(t))dt$$

for all $x \in [0,1]$. (Hint: first show that $|f_n(x_1) - f_n(x_2)| \leq |x_1 - x_2|$.)

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