Ph.D. Qualifying Examination
Sem 1, 2002/2003
Complex Analysis

1. Consider the function $f(z) = 5\overline{z}^2 - 2\overline{z} - 3i|z|^2$ in the complex plane. Determine the points, if any, at which $f(z)$ is differentiable, and find $f'(z)$ where it exists. Determine also the points, if any, at which $f(z)$ is analytic.

2. Let $h(x, y)$ be a harmonic function in $\mathbf{R}^2$ such that $h(x, y) \neq 0$ for any $(x, y) \in \mathbf{R}^2$. Consider the function $g(x, y) = \dfrac{1}{[h(x, y)]^2}$ in $\mathbf{R}^2$. Suppose that $g(x, y)$ is also harmonic in $\mathbf{R}^2$. Is it true that $g(x, y)$ must be a constant function? Justify your answer.

3. Use the residue theorem to eveluate the improper integral
$$\int_0^\infty \frac{\cos 2x}{(x^2 + 9)^2}\, dx.$$
Justify your steps.

4. Suppose that a function q is analytic and has a simple zero at a point z_o (that is, $q(z_o) = 0$ and $q'(z_o) \neq 0$). Consider the function
$$f(z) = \frac{1}{(z - z_o)^2 q(z)}.$$
Show that f has a pole of order 3 at z_o, and express the residue of f at z_o in terms of $q'(z_o)$, $q''(z_o)$ and $q'''(z_o)$. Justify your answer.

5. Let f be an entire function such that $|f(z)| = 1$ for any complex number z satisfying $|z| = 1$. Prove that there exist a non-negative integer n and a complex number c satisfying $|c| = 1$ such that
$$f(z) = cz^n$$
for all z in the complex plane.

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