Ph.D. Qualifying Examination
Sem 1, 2002/2003
Analysis

1.(a) Let $f : [0, \infty) \to \mathbb{R}$. Suppose that f is continuous on $[0, \infty)$ and differentiable on $[100, \infty)$ with bounded derivatives there. Prove that f is uniformly continuous on $[0, \infty)$.

(b) Let $f : (0, 1] \to \mathbb{R}$ be continuous. Is f uniformly continuous on $(0, 1]$? Justify your answer.

2.(a) State, without proof, the Heine-Borel Theorem.

(b) Let δ be a positive function defined on $[a, b]$. Prove that there exist a finite number of interval-point pairs $([u_i, v_i], x_i)$, with $x_i \in [u_i, v_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i))$, $i = 1, 2, \ldots, n$, satisfying the following properties:

(i) $(u_i, v_i) \cap (u_j, v_j) = \phi$ for $i \neq j$;

(ii) $x_i \in [u_i, v_i]$ for each i; and

(iii) $\displaystyle\bigcup_{i=1}^{n} [u_i, v_i] = [a, b]$.

3.(a) Let $f : [a, b] \to \mathbb{R}$. Suppose that f is unbounded on $[a, b]$. Prove that there exists a convergent sequence $\{y_n\}$ in $[a, b]$ such that $|f(y_n)| > n$, for each n.

(b) Use (a) to prove that if f is continuous on $[a, b]$, then f is bounded on $[a, b]$.

4. Let $\{f_n\}$ and $\{g_n\}$ be two sequences of functions defined on $[a, b]$. Suppose that

(i) $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$;

(ii) $g_n(x) \leq g_{n+1}(x)$ for all $x \in [a, b]$ and all n; and

(iii) there exists a real number L such that $|g_n(x)| \leq L$ for all $x \in [a, b]$ and all n.

Prove that $\displaystyle\sum_{n=1}^{\infty} f_n(x) g_n(x)$ converges uniformly on $[a, b]$.

Hint: Use Cauchy Criterion and Abel's partial summation

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) B_k + a_n B_n$$

where $B_k = \displaystyle\sum_{i=1}^{k} b_i$.

5. Let $C^*[0,1]$ be the space of all functions $x : [0,1] \to [0,1]$, which are continuous and $x(0) = 0$. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. For each $x \in C^*[0,1]$, define $F(x) : [0,1] \to \mathbb{R}$ as follows:

$$F(x)(t) = \int_0^t f(s, x(s))ds \text{ for } t \in [0,1].$$

Let $G = \{F(x) : x \in C^*[0,1]\}$. Prove that

(i) G is sequentially compact i.e., every sequence in G has a subsequence which is uniformly convergent on $[0,1]$;

(ii) $F : C^*[0,1] \to C[0,1]$ is continuous under the uniform norm $\| \ \|$, where $C[0,1]$ is the space of all continuous functions on $[0,1]$ and $\|x\| = \sup\{x(t) : t \in [0,1]\}$.

— END OF PAPER —