Ph.D. Qualifying Examination
Sem 2, 2001/2002
Linear Algebra

1. Let $T : V \to V$ be a linear operator on a finite dimensional vector space V. Let W be the subspace of V spanned by $\{v, T(v), T^2(v), \ldots\}$ where v is some nonzero vector. Show that there exists a positive integer k such that $\{v, T(v), T^2(v), \ldots, T^k(v)\}$ form a basis for W.

2. Let A be a square matrix. Show that A and A^T have the same characteristic polynomial and minimum polynomial.

3. Let $T : V \to V$ be a linear operator on V and $\{v_1, v_2, v_3, v_4\}$ be a basis for V. Suppose $T(v_1) = -2T(v_2)$, $T(v_2) = v_1$, $T(v_3) = v_4$, $T(v_4) = v_3$. Is T diagonalisable?

4. Let $u, v \in V$, a real vector space with inner product $< , >$.
   Show that $| < u, v > | = \|u\|\|v\|$ if and only if u, v are linearly dependent.
   ($\|u\|$ is the norm of the vector u with respect to the inner product $< , >$.)

5. Let B_1 and B_2 be two orthonormal bases for a finite dimensional real inner product space V. Show that the transition matrix P_{B_2,B_1} from B_1 to B_2 is an orthogonal matrix.
   (P_{B_2,B_1} is the square matrix such that $P_{B_2,B_1}[v]_{B_1} = [v]_{B_2}$ for all $v \in V$ and $[v]_B$ denotes the coordinate vector of v with respect to the basis B.)

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