Ph.D. Qualifying Examination
Sem 2, 2001/2002
Complex Analysis

1. Let
$$f(x+iy) = (x^3 - 9x^2 + y^3) - i(3x + 24y).$$
Determine the points, if any, at which f is differentiable, and find $f'(z)$ where it exists. Determine also the points, if any, at which f is analytic.

2. Suppose that f is an entire function and $f(z) = f(2z)$ for all complex numbers z. Prove that f is a constant function.

3. Let
$$f(z) = \frac{1}{(z-1)(z-3)}.$$
(i) Find the Laurent series of f in the domain $\{z \in \mathbb{C} :\ 1 < |z| < 3\}$.
(ii) Evaluate the integral
$$\int_\gamma \frac{1}{z^n(z-1)(z-3)}$$
where n is a positive integer and γ is the positively oriented circle $|z| = 2$.

4. Let
$$f(z) = z\left(\frac{z-2}{e^{i\pi z} - 1}\right)^2.$$
(i) Identify the singular points of f and classify each as a removable singularity, an essential singularity or a pole of specific order.
(ii) Evaluate
$$\int_\gamma f(z)\ dz$$
where γ is the positively oriented circle $|z| = 1$.

5. Use the Cauchy residue theorem to show that
$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{\pi(2n)!}{2^{2n}(n!)^2}$$
for all nonnegative integers n.

— END OF PAPER —