Ph.D. Qualifying Examination
Sem 2, 2001/2002
Analysis

1. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers such that $a_n \neq 0$ for all $n \in \mathbb{N}$.

   (a) Show that
   $$\limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} |a_{n+1}/a_n|.$$

   (b) If $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists, show that $\lim_{n\to\infty} |a_n|^{1/n}$ exists and the two limits are equal.

   (c) Give an example where equality does not hold in (a).

2. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers.

   (a) Define $b_0 = 0$ and $c_n = b_n - b_{n-1}$ for all $n \in \mathbb{N}$. Show that if $p, q \in \mathbb{N}$, $p \leq q$, then
   $$\sum_{n=p}^{q} a_n b_n = \left(\sum_{n=p}^{q} a_n\right) b_{p-1} + \sum_{j=p}^{q} \left(\sum_{n=j}^{q} a_n\right) c_j.$$

   (b) Suppose that $(b_n)_{n=1}^{\infty}$ is increasing and converges to $b \in \mathbb{R}$, and that $\sum_{n=1}^{\infty} a_n$ converges. Let M and m be real numbers such that $m \leq \sum_{n=p}^{q} a_n \leq M$ for all p, $q \in \mathbb{N}$, $p \leq q$. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges and that $mb \leq \sum_{n=1}^{\infty} a_n b_n \leq Mb$.

   (c) If $\sum_{n=1}^{\infty} a_n x^n$ converges for all $x \in [0, 1]$, show that
   $$\lim_{x\to 1^-} \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n.$$

3. Let $f : X \to Y$ be a function mapping between metric spaces X and Y. Show that f is continuous on X if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X. (Here $\overline{S}$ denotes the closure of the set S.)

4. Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions from a metric space X into a metric space Y. If $(f_n)_{n=1}^{\infty}$ converges uniformly to a function f on X and $(x_n)_{n=1}^{\infty}$ is a sequence in X that converges to an element $x \in X$, show that $(f_n(x_n))_{n=1}^{\infty}$ converges to $f(x)$.

5. Let $f : (0, 1] \to \mathbb{R}$ be a continuous function on $(0, 1]$. Show that f is uniformly continuous on $(0, 1]$ if and only if $\lim_{x\to 0^+} f(x)$ exists and has a real value.

— END OF PAPER —