Ph.D. Qualifying Examination
Sem 2, 2001/2002
Algebra

1. Classify all groups of order n up to isomorphism, where

   (a) n is the square of a prime integer; [10 marks]

   (b) $n = pq$ where p, q are primes with $p > q$ and q does not divide $p - 1$. [10 marks]

2. Let R be a commutative ring with 1.

   (a) If $I, J_1, J_2, \ldots, J_n$ are ideals of R such that I is prime and $I \supseteq \bigcap_{r=1}^{n} J_r$, prove that $I \supseteq J_s$ for some s. [10 marks]

   (b) If the intersection of all maximal ideals of R is prime but not maximal, prove that R has infinitely many maximal ideals. [5 marks]

   (c) If $I, J_1, J_2, \ldots, J_n$ are ideals of R such that J_r's are prime for all r, and $I \subseteq \bigcup_{r=1}^{n} J_r$, prove that $I \subseteq J_s$ for some s. [15 marks]

3. Let R be a ring and M be a left R-module. Show that the following statements are equivalent: [30 marks]

   (a) Every submodule of M is finitely generated.

   (b) Every non-empty collection of submodules of M has a maximal element (with respect to inclusion).

   (c) Whenever $N_1 \subseteq N_2 \subseteq \cdots$ is an ascending chain of submodules of M, there is an integer k such that $N_l = N_k$ for all $l \geq k$.

4. A complex number is *algebraic* if and only if it satisfies a polynomial with rational coefficients.

   (a) Prove that a complex number α is algebraic if and only if $\alpha \in F$ for some finite field extension F of $\mathbb{Q}$. [8 marks]

   (b) Hence, or otherwise, show that the set K of algebraic numbers is a field. [6 marks]

   (c) Show further that K is algebraically closed, i.e. every polynomial with coefficients in K has a root in K. [6 marks]

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