Ph.D. Qualifying Examination
Sem 1, 2001/2002
Complex Analysis

1. Find and classify all the isolated singular points of the following functions. Evaluate also the residues of the functions at the isolated singular points.
   (i) $\dfrac{z - \cos(\pi z)}{(z^2 - 1)^3}$;
   (ii) $(z^2 + 4z)e^{\frac{1}{z-1}}$.

2. Determine the smallest positive integer n such that there is no complex number z satisfying
$$z^{10} + z^5 + 20z + 2001 = 0 \quad \text{and} \quad |z| \geq n.$$
   Justify your answer.

3. Use the residue theorem to eveluate the improper integral
$$\int_0^{\infty} \frac{x \sin 3x}{x^4 + 64}\, dx.$$
   Justify your steps.

4. Let $f(z)$ be an analytic function on the complex plane $\mathbf{C}$ such that $|f(z)| < 1$ for all $|z| < 1$.
   (i) For a complex number a such that $|a| < 1$, define the function
$$g(z) = \frac{f(z) - a}{1 - \bar{a} f(z)}.$$
   Show that $|g(z)| < 1$ for all $|z| < 1$.
   (ii) Hence or otherwise, show that
$$\frac{|f(0)| - |z|}{1 - |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)||z|}.$$

5. Let f be an analytic function in the complex plane $\mathbf{C}$. Show that the real part $\mathrm{Re}(f)$ of f cannot have an isolated zero. (An *isolated zero* of a function u is a point $z_o \in \mathbf{C}$ such that $\{z \in V : u(z) = 0\} = \{z_o\}$ for some open neighborhood V of z_o.)

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