Ph.D. Qualifying Examination
Sem 1, 2001/2002
Analysis

1. Let d be a metric on a nonempty set M. For each of the following, determine whether in general ρ defines a metric on M. Justify your answers.
   (i) $\rho(x,y) = (d(x,y))^2$, $x, y \in M$.
   (ii) $\rho(x,y) = \min\{2, d(x,y)\}$, $x, y \in M$.

2. Prove or disprove the following statements.
   (a) In a metric space, every closed subset of a compact set is compact.
   (b) In a metric space, every closed and bounded set is compact.

3. Let ℓ^∞ be the space of all bounded sequences of complex numbers endowed with the metric
$$d(\zeta, \eta) = \sup_{j\in\mathbb{N}} |\zeta_j - \eta_j|, \qquad \zeta = \{\zeta_j\}_{j\in\mathbb{N}}, \eta = \{\eta_j\}_{j\in\mathbb{N}} \in \ell^\infty.$$
Suppose that $K : \mathbb{N}^2 \longrightarrow \mathbb{C}$ is a function for which there exists $\lambda \in (0,1)$ such that
$$\sum_{l\in\mathbb{N}} |K(j,l)| \le \lambda, \qquad j \in \mathbb{N}.$$
Show that for every $\beta = \{\beta_j\}_{j\in\mathbb{N}} \in \ell^\infty$, there exists a unique $\alpha = \{\alpha_j\}_{j\in\mathbb{N}} \in \ell^\infty$ such that
$$\alpha_j = \sum_{l\in\mathbb{N}} K(j,l)\alpha_l + \beta_j, \qquad j \in \mathbb{N}.$$

4. Determine whether the function $g(x,y) = \displaystyle\sum_{k=1}^{\infty} \frac{(x-2y)^k \sin(kx+y)}{\sqrt{k!}\,(1+x^{2k}y^{4k})}$ is continuous on $\mathbb{R}^2$. Justify your answer.

5. Let $f_k : [0,1] \longrightarrow \mathbb{R}$, $k \ge 1$, be a sequence of continuous functions such that for every $k \ge 1$,
$$\int_0^1 (f_k(t))^2\, dt = 1.$$
Define a sequence of functions $F_k : [0,1] \longrightarrow \mathbb{R}$, $k \ge 1$, by
$$F_k(x) = \int_0^x t f_k(t)\, dt.$$
Prove that the sequence F_k, $k \ge 1$, has a uniformly convergent subsequence.

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