NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2000/2001

MA1101 Linear Algebra I

October/November 2000 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in **Section A**. The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [15 marks]

(a) Given that $A = \begin{pmatrix} 2 & 3 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ -4 & 3 \end{pmatrix}$. Determine a matrix X satisfying

$$A^{-1}XA = B.$$

(b) Let A, B be 4×4 matrices. Suppose that $B = E_3\, E_2\, E_1\, A$, where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Write down a sequence of elementary row operations, which when applied to A, gives matrix B.

(ii) Suppose that $B = I$, the 4×4 identity matrix.
Find elementary matrices E, F, G such that $A = EFG$.

Question 2 [15 marks]

(a) Let V be a real vector space and let $w_1, w_2, \cdots, w_n$ be vectors in V. Suppose that a vector u in V is a linear combination of $w_1, w_2, \cdots, w_n$.

Show that if $w_1, w_2, \cdots, w_n$ are linearly dependent, then u can be expressed as a linear combination of $w_1, w_2, \cdots, w_n$ in infinitely many ways.

(b) Let V be the vector space of all 2×2 real matrices. Let W be the subspace consisting of all the symmetric matrices.

Show that the set $S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ forms a basis of W.

Give reasons why $T = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ is not a basis of W.

Question 3 [15 marks]

(a) Let $M_{2\times 2}$ be the vector space of 2×2 matrices over $\mathbb{R}$. Determine whether each of the following subsets of $M_{2\times 2}$ is a subspace of $M_{2\times 2}$.

(i) $S = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a - b + c = 0\}$.

(ii) $T = \{A \mid$ the linear homogeneous system $AX = 0$ has infinitely many solutions$\}$.

(b) Let $u = (1,2,3)$, $v = (2,0,-1)$, $w = (3,2,2)$ be vectors of $\mathbb{R}^3$.

Show that $\{u, v, w\}$ does not span $\mathbb{R}^3$; and give an example of a vector which lies in $\mathbb{R}^3$ but not in the subspace spanned by $\{u, v, w\}$.

Question 4 [15 marks]

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longmapsto \begin{bmatrix} c \\ a+b+c \\ b+c+d \end{bmatrix} .$$

(i) Determine the standard matrix of T.

(ii) Determine bases for the range and the kernel of T respectively.

SECTION B

Answer not more than **two** *questions in this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

(a) Let A and B be $n\times n$ matrices. Consider each of the following statements. If it is true, give a proof; if it is false, give a counter example.

(i) If $A^2 = B^2$, then $A = B$ or $A = -B$.

(ii) If A is symmetric and invertible, then the inverse A^{-1} of A is also symmetric.

(iii) If the product AB is invertible, then both A and B are invertible.

Question 5 (continued)

(b) Determine whether each of the following mappings is a linear transformation.

(i) $T : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{bmatrix} a \\ b \end{bmatrix} \longmapsto a + b.$$

(ii) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto \begin{bmatrix} a^2 \\ b \end{bmatrix}.$$

Question 6 [20 marks]

(a) Let V be a real vector space and let u, v, w be linearly independent vectors of V. Consider the subspace $W = span(u - v + w, u - w, 2u - v)$.

(i) Show that $\{u - v + w, u - w, 2u - v\}$ does not form a basis of W.

(ii) Find a basis of W. Justify your answer.

(b) Let $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ -2 & 1 & 1 & 2 \\ 0 & 1 & 3 & 6 \\ 1 & 1 & 4 & 8 \end{pmatrix}$.

(i) Determine the rank of A.

(ii) Find a basis for the row space of A; and extend this basis to form a basis of $\mathbb{R}^4$.

Question 7 [20 marks]

(a) Consider the line

$$\{(1, 3, 0) + t(2, 1, 1) | t \in \mathbb{R}\}$$

in the Euclidean space $\mathbb{R}^3$.

Find equations of three distinct planes in $\mathbb{R}^3$ which will meet at the above line.

(b) Let V, W be real vector spaces and let $T : V \to W$ be a linear transformation.

(i) Denote by 0_V and 0_W the zero vectors of V and W respectively. Show that $T(0_V) = 0_W$.

(ii) Suppose that $x_1, x_2, \cdots, x_n$ are linearly independent vectors of V and that the kernel ker (T) of T is the zero subspace of V. Show that the vectors $T(x_1)$, $T(x_2)$, $\ldots, T(x_n)$ of W are linearly independent.

END OF PAPER