Ph.D. Qualifying Examination
Sem 2, 2000/2001
Analysis

1. Show that a function $f : [0,1] \to \mathbb{R}$ is uniformly continuous on $[0,1]$ if and only if $(f(x_n))_{n=1}^{\infty}$ is a Cauchy sequence whenever $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $[0,1]$.

2. Let f be a continuous real-valued function such that

$$\int_0^1 x^n f(x)\,dx = 0 \text{ for } n = 0, 1, 2, \ldots$$

Show that f is the zero function.

3. Let $C[0,1]$ denote the set of all real-valued continuous functions on $[0,1]$. Show that there is a unique $f \in C[0,1]$ such that

$$f(x) = \int_0^{x/2} f(t)\,dt$$

for all $x \in [0,1]$.

4. Let (X,d) be a complete metric space. For any $x \in X$ and any $\epsilon > 0$, let $B(x,\epsilon)$ denote the open ball of radius ϵ centered at x. Suppose that A is a subset of X so that for any $\epsilon > 0$, there exists a compact subset A_ϵ of X satisfying

$$A \subseteq \cup_{x \in A_\epsilon} B(x,\epsilon).$$

Show that A is relatively compact, i.e., the closure of A is a compact set.

5. Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued differentiable functions on $[0,1]$. Assume that there is a constant $C < \infty$ so that
   (i) $|f_n(0)| \le C$ for all $n \in \mathbb{N}$;
   (ii) $|f_n'(x)| \le C$ for all $x \in [0,1]$ and all $n \in \mathbb{N}$. Show that $(f_n)_{n=1}^{\infty}$ has a uniformly convergent subsequence.

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