NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 1999/2000

MA1101 LINEAR ALGEBRA I

April/May 2000 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions in **Section A**. Each question in Section A carries 15 marks.

3. Answer not more than **TWO** questions from **Section B**. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [15 Marks]

(a) Consider the following system of linear equations

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0. \end{cases}$$

Find the general solution by Gaussian or Gauss-Jordan Elimination.

(b) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 2 & 10 & 9 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 2 & -3 \end{pmatrix}$. Find elementary matrices $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ such that $\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{B}$ and $\boldsymbol{E}_2\boldsymbol{B} = \boldsymbol{A}$.

(c) List all possible reduced row echelon forms of a 3×3 homogeneous linear system whose solution set represents a line in the three dimensional space.

Question 2 [15 Marks]

Let $S = \{(3,2,0,2), (12,5,0,2), (6,2,5,2), (3,2,0,5)\}$ be a subset of $\mathbb{R}^4$.

(i) Show that S is linearly independent.

(ii) Is span(S) equal to $\mathbb{R}^4$? Justify your answer.

(iii) If we are to form a basis for the subspace $W = \{(x,y,0,z) \mid x,y,z \in \mathbb{R}\}$ using vectors from S, which of the vectors shall we throw away? Explain why the remaining vectors in S form a basis for W.

(iv) Write down a subspace of W whose dimension is 2.

Question 3 [15 Marks]

(a) Let $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 0 & 1 & 0 \\ 2 & -4 & 2 & 4 & 6 \\ 3 & -6 & 1 & 4 & 1 \\ 1 & -2 & -2 & -1 & -3 \end{pmatrix}$.

(i) Find a basis for each of the row space, column space and nullspace of the matrix $\boldsymbol{A}$. Show your workings clearly.

(ii) Verify the Dimension Theorem for the matrix $\boldsymbol{A}$.

(b) Given that $\boldsymbol{B}$ is a 5×3 matrix whose rank is 3, find rank($\boldsymbol{B}^{\boldsymbol{T}}$), nullity($\boldsymbol{B}$) and nullity($\boldsymbol{B}^{\boldsymbol{T}}$).

Question 4 [15 Marks]

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y-z \\ x+z \end{pmatrix}$.

(i) Show that T is a linear transformation.

(ii) Find the standard matrix of T.

(b) Let $F : \mathbb{R}^3 \to M_{2\times 2}$ be a linear transformation such that $F(1,1,0) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $F(1,0,1) = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$ and $F(0,1,1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
Find $F(1,1,1)$.

(c) For each of the following cases, find the transformation matrix of

(i) the scaling with factor 1/2 followed by reflection about the line $y = x$ in $\mathbb{R}^2$;

(ii) the projection onto the xz-plane in $\mathbb{R}^3$.

SECTION B

Answer not more than **two** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 Marks]

(a) Let $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & a+3 & 0 & a-3 \\ 0 & 0 & a^2-9 & a-3 \end{array}\right)$ be the augmented matrix of a linear system where a is some real number. Find all the possible values of a such that the system has (i) no solution; (ii) a unique solution; (iii) infinitely many solutions. Justify your answer.

(b) A square matrix $\boldsymbol{B}$ is said to be *nilpotent* if $\boldsymbol{B}^k = \boldsymbol{0}$ for some positive integer k.

(i) Show that $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent.

(ii) Show that, if a square matrix $\boldsymbol{C}$ is nilpotent, then $\boldsymbol{C}$ is not invertible.

(c) Let $\boldsymbol{A} = (a_{ij})_{n\times n}$ be an $n \times n$ matrix with real entries.

(i) Write down the (i, j)-entry of $\boldsymbol{A}\boldsymbol{A}^T$.

(ii) If $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{0}$, show that $\boldsymbol{A} = \boldsymbol{0}$.

Question 6 [20 Marks]

(a) Show that

(i) $W = \{(a, b, c, d) \mid a = c \text{ and } b = d\}$ is a subspace of $\mathbb{R}^4$.

(ii) $S = \{\boldsymbol{A} \in M_{2\times 2} \mid \boldsymbol{A} \text{ is not invertible}\}$ is not a subspace of $M_{2\times 2}$.

(b) True or false: Let S_1 and S_2 be two subsets of a vector space. Then

$$\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2).$$

Justify your answer.

(c) Let $S = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$ be a basis for a vector space V

(i) Given a vector $\boldsymbol{v}$ in V, describe what do we mean by $(\boldsymbol{v})_S$, the coordinate vector of $\boldsymbol{v}$ with respect to S.

(ii) Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent if and only if $\{(\boldsymbol{v}_1)_S, (\boldsymbol{v}_2)_S, (\boldsymbol{v}_3)_S\}$ is linearly independent.

Question 7 [20 Marks]

(a) Let $T : M_{2\times 2} \to M_{2\times 2}$ be a linear transformation defined by $T(\boldsymbol{A}) = \boldsymbol{A} - \boldsymbol{A}^T$.

(i) Describe the range $R(T)$ and the kernel $\ker(T)$ of T in set notation form.

(ii) Find the rank and nullity of T. Justify your answer.

(b) Let $\boldsymbol{A}$ be the standard matrix of a linear transformation $T_{\boldsymbol{A}}$ on $\mathbb{R}^n$. Prove that $\boldsymbol{A}$ is invertible if and only if $R(T_{\boldsymbol{A}}) = \mathbb{R}^n$.

(c) Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ column vector. Show that, if the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution, then the nullity of $\boldsymbol{A}$ is zero. Is the converse true?

[END OF PAPER]