Matriculation Number:

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# MA1505 Practice Examination

**VERY IMPORTANT: This practice examination is just a sample for the students' reference only. Students should NOT expect to find all the questions in the actual examination to be of exactly the same type as the questions in the practice examination. The level of difficulty of the actual examination may also be DIFFERENT from the level of difficulty of the practice examination.**

## INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

**For official use only. Do not write below this line.**

| Question | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| (a) | | | | | | | | |
| (b) | | | | | | | | |

## Question 1 (a) [5 marks]

Let $f(x)$ be a function defined by

$$f(x) = x^{1505} \quad \text{if } -\pi < x < \pi\,,$$

and $f(x + 2\pi) = f(x)$.
Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series for $f(x)$.
Find the **exact value** of $a_{2012}$.

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| Answer 1(a) | $0$ |
|---|---|

*(Show your working below and on the next page.)*

$\because f$ is an odd function

$\therefore a_n = 0$ for all $n$.

$\therefore a_{2012} = \underline{\underline{0}}$

**Question 1 (b)** [5 marks]

Let $f(x)$ be a function defined by

$$f(x) = |\sin x| \quad \text{if} \ -\pi < x < \pi \ ,$$

and $f(x + 2\pi) = f(x)$.
Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series for $f(x)$.
Find the **exact value** of $a_4$.

| **Answer 1(b)** | $-\frac{4}{15\pi}$ |
|---|---|

*(Show your working below and on the next page.)*

Note that $f$ is an even function

$(\because f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x))$

$$\therefore a_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 4x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos 4x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos 4x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (\sin 5x - \sin 3x)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{5}\cos 5x + \frac{1}{3}\cos 3x\right]_0^{\pi}$$

$$= \underline{\underline{-\frac{4}{15\pi}}}$$

**Question 2 (a)** [5 marks]

Let $f(x)$ be a function defined on the open interval $(0, 1)$ by

$$f(x) = x \quad .$$

Find the first two non-zero terms of the cosine Fourier half range expansion for $f(x)$.

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| **Answer 2(a)** | $\frac{1}{2} - \frac{4}{\pi^2}\cos \pi x$ |
|---|---|

*(Show your working below and on the next page.)*

$$a_0 = \frac{1}{1}\int_0^1 x\,dx = \frac{1}{2}$$

$$a_n = \frac{2}{1}\int_0^1 x \cos n\pi x\,dx$$

$$= \frac{2}{n\pi}\int_0^1 x\,d(\sin n\pi x)$$

$$= \frac{2}{n\pi}\left\{\left[x \sin n\pi x\right]_0^1 - \int_0^1 \sin n\pi x\,dx\right\}$$

$$= \frac{2}{n^2\pi^2}\cos n\pi x\Big|_0^1$$

$$= \frac{2}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

$$a_1 = -\frac{4}{\pi^2}$$

$$\therefore f \sim \frac{1}{2} - \frac{4}{\pi^2}\cos \pi x + \cdots$$

**Question 2 (b)** [5 marks]

Let $w = f(x, y, z)$ be a differentiable function of three variables and

$$x = u + v, \quad y = u - v, \quad z = u^2 + v^2.$$

Use the Chain Rule to find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ in terms of $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$.

| | |
|---|---|
| **Answer 2(b)** | $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + 2u\frac{\partial w}{\partial z}$ <br> $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} + 2v\frac{\partial w}{\partial z}$ |

*(Show your working below and on the next page.)*

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + 2u\frac{\partial w}{\partial z}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$= \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} + 2v\frac{\partial w}{\partial z}$$

## Question 3 (a) [5 marks]

The temperature at a point $(x, y, z)$ in space is given by

$$T(x,y,z) = 25 + \left(1 - \frac{z}{100}\right) e^{-\left(x^2+y^2\right)}.$$

Find the unit vector of the direction in which the temperature increases most rapidly at the point $(2, 0, 99)$.

| | |
|---|---|
| **Answer 3(a)** | $-\frac{4}{\sqrt{17}}\vec{i} - \frac{1}{\sqrt{17}}\vec{k}$ |

*(Show your working below and on the next page.)*

$$\nabla T = \left(1 - \frac{z}{100}\right) e^{-(x^2+y^2)}(-2x)\,\vec{i} + \left(1-\frac{z}{100}\right) e^{-(x^2+y^2)}(-2y)\,\vec{j} - \frac{1}{100} e^{-(x^2+y^2)}\,\vec{k}$$

$$\nabla T(2,0,99) = \frac{-4}{100} e^{-4}\,\vec{i} + 0\vec{j} - \frac{1}{100} e^{-4}\,\vec{k} = \frac{e^{-4}}{100}\left(-4\vec{i} - \vec{k}\right)$$

unit vector $\vec{u} = \dfrac{-4\vec{i} - \vec{k}}{\sqrt{16+1}} = \underline{\underline{-\frac{4}{\sqrt{17}}\vec{i} - \frac{1}{\sqrt{17}}\vec{k}}}$

## Question 3 (b) [5 marks]

A rocket is launched with a constant thrust corresponding to an acceleration of $u$ feet per second squared. Ignoring air resistance, the rocket's height after $t$ seconds is given by

$$h(t,u) = \frac{1}{2}(u-32)\,t^2 \text{ feet.}$$

Fuel usage for $t$ seconds is proportional to $u^2 t$ and the limited fuel capacity of the rocket satisfies the equation $u^2 t = 10000$. Find the value of $u$ that maximizes the height that the rocket reaches when the fuel runs out.

| Answer 3(b) | $\frac{128}{3}$ |
|---|---|

*(Show your working below and on the next page.)*

$h(t,u) = \frac{1}{2}(u-32)t^2 = \max!$

$g(t,u) = u^2 t - 10000 = 0$

Let $F(t,u) = \frac{1}{2}(u-32)t^2 + \lambda(u^2 t - 10000)$

$F_t = 0 \Rightarrow (u-32)t + \lambda u^2 = 0$ ---- ①

$F_u = 0 \Rightarrow \frac{1}{2}t^2 + 2\lambda u t = 0$ ---- ②

① $\times 2t \Rightarrow 2(u-32)t^2 + 2\lambda t u^2 = 0$ ---- ③

② $\times u \Rightarrow \frac{1}{2}ut^2 + 2\lambda t u^2 = 0$ ---- ④

③ & ④ $\Rightarrow 2(u-32)t^2 = \frac{1}{2}ut^2$

$\Rightarrow 4u - 128 = u \quad (\because u^2 t = 10000 \therefore t \neq 0)$

$\Rightarrow u = \frac{128}{3}$

## Question 4 (a) [5 marks]

Find the value of the double integral

$$\int\int_D (2xy^2 + 2y\cos x)\,dxdy,$$

where $D$ is the finite domain bounded by the curve $y = \sqrt{x}$ and the two lines: $x = 0$, $y = 3$. Give your answer correct to the nearest integer.

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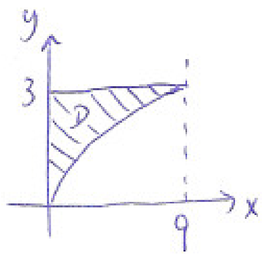
| Answer 4(a) | 314 |
|---|---|

*(Show your working below and on the next page.)*




$$\iint_D (2xy^2 + 2y\cos x)\,dxdy$$

$$= \int_0^3 \int_0^{y^2} (2xy^2 + 2y\cos x)\,dx\,dy$$

$$= \int_0^3 \left[x^2 \cdot y^2 + 2y\sin x\right]_{x=0}^{x=y^2} dy$$

$$= \int_0^3 (y^6 + 2y\sin y^2)\,dy$$

$$= \left[\tfrac{1}{7}y^7 - \cos y^2\right]_0^3$$

$$= \tfrac{1}{7}(3)^7 - \cos 9 + 1$$

$$= 314.339\cdots$$

$$\approx \underline{\underline{314}}$$

## Question 4 (b) [5 marks]

Find the **exact value** of the iterated integral

$$\int_0^1 \int_y^1 3xe^{x^3}\,dx\,dy.$$

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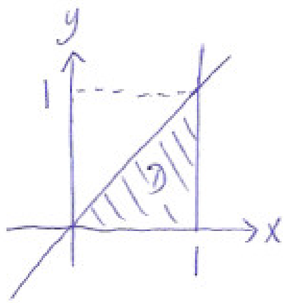
| | |
|---|---|
| **Answer 4(b)** | $e-1$ |

*(Show your working below and on the next page.)*




$$\int_0^1 \int_y^1 3xe^{x^3}\,dx\,dy$$

$$= \int_0^1 \int_0^x 3xe^{x^3}\,dy\,dx$$

$$= \int_0^1 3x^2e^{x^3}\,dx$$

$$= e^{x^3}\Big|_0^1$$

$$= \underline{\underline{e-1}}$$

## Question 5 (a) [5 marks]

Find the **exact value** of the volume of the tetrahedron bounded by the plane $2x + y + z = 2$ and the three coordinate planes.

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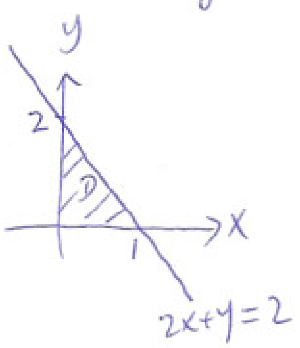
| **Answer 5(a)** | $\frac{2}{3}$ |

*(Show your working below and on the next page.)*

$z = 0 \Rightarrow 2x + y = 2$




$$\text{Vol} = \iint_D (2 - 2x - y)\, dx\, dy$$

$$= \int_0^1 \int_0^{2-2x} (2 - 2x - y)\, dy\, dx$$

$$= \int_0^1 \left[ (2-2x)y - \tfrac{1}{2}y^2 \right]_{y=0}^{y=2-2x} dx$$

$$= \tfrac{1}{2} \int_0^1 (2 - 2x)^2\, dx$$

$$= 2 \int_0^1 (1 - 2x + x^2)\, dx$$

$$= 2 \left[ x - x^2 + \tfrac{1}{3}x^3 \right]_0^1$$

$$= \underline{\underline{\tfrac{2}{3}}}$$

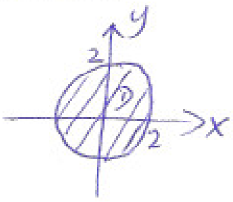
## Question 5 (b) [5 marks]

Find the **exact value** of the surface area of that portion of the paraboloid $z = 1 + x^2 + y^2$ that lies below the plane $z = 5$.

| | |
|---|---|
| **Answer 5(b)** | $\frac{\pi}{6}\left(17^{3/2} - 1\right)$ |

*(Show your working below and on the next page.)*

$z = 5 \Rightarrow x^2 + y^2 = 4 \Rightarrow$




$z_x = 2x$

$z_y = 2y$

$$\text{surface area} = \iint_D \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy$$

$$= \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= 2\pi \int_0^2 (1 + 4r^2)^{1/2}\, d(1 + 4r^2)/8$$

$$= \frac{\pi}{4} \frac{2}{3} (1 + 4r^2)^{3/2} \Big|_0^2$$

$$= \underline{\underline{\frac{\pi}{6}\left(17^{3/2} - 1\right)}}$$

## Question 6 (a) [5 marks]

Find the **exact value** of the line integral

$$\int_C 2y\,ds,$$

where $C$ is the space curve given by

$$x = 2\cos t, \quad y = t, \quad z = 2\sin t,$$

with $0 \leq t \leq 6\pi$.

| **Answer 6(a)** | $36\sqrt{5}\,\pi^2$ |
|---|---|

*(Show your working below and on the next page.)*

$\vec{r}(t) = (2\cos t,\ t,\ 2\sin t)$

$\vec{r}\,'(t) = (-2\sin t,\ 1,\ 2\cos t)$

$\|\vec{r}\,'(t)\| = \sqrt{5}$

$\int_C 2y\,ds = \int_0^{6\pi} 2t\sqrt{5}\,dt$

$= \sqrt{5}\left[t^2\right]_0^{6\pi}$

$= \underline{\underline{36\sqrt{5}\,\pi^2}}$

**Question 6 (b)** [5 marks]

Find the **exact value** of the line integral

$$\int_C 4x\,dy + 2y\,dz,$$

where $C$ is the space curve consists of the line segment from $(0,1,0)$ to $(0,1,1)$, followed by the line segment from $(0,1,1)$ to $(2,1,1)$ and followed by the line segment from $(2,1,1)$ to $(2,4,1)$.

| Answer 6(b) | 26 |
| --- | --- |

*(Show your working below and on the next page.)*

$C_1$ : $x=0,\ y=1,\ z=t,\ 0 \le t \le 1$

$C_2$ : $x=t,\ y=1,\ z=1,\ 0 \le t \le 2$

$C_3$ : $x=2,\ y=t,\ z=1,\ 1 \le t \le 4$

$$\int_C 4x\,dy + 2y\,dz = \int_{C_1+C_2+C_3} (4x\,dy + 2y\,dz)$$

$$= \int_0^1 2\,dt + \int_0^2 0 + \int_1^4 8\,dt$$

$$= 2 + 24$$

$$= \underline{\underline{26}}$$

## Question 7 (a) [5 marks]

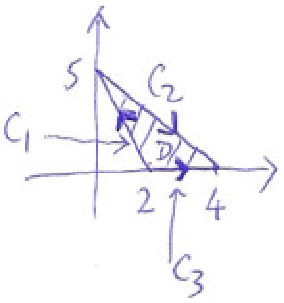
Find the **exact value** of the line integral

$$\int_{C_1+C_2} (xe^{x^2} + 2y)dx + (3x + e^{y\sqrt{1+y^2}})dy,$$

where $C_1$ is the straight line segment that joins $(2,0)$ to $(0,5)$ and $C_2$ is the straight line segment that joins $(0,5)$ to $(4,0)$.

| | |
|---|---|
| **Answer 7(a)** | $\frac{1}{2}e^{16} - \frac{1}{2}e^{4} - 5$ |

*(Show your working below and on the next page.)*




Let $C_3 : x = t,\ y = 0,\ 2 \le t \le 4$

Green's Theorem

$$\Rightarrow \oint_{C_1+C_2-C_3} (xe^{x^2}+2y)\,dx + (3x+e^{y\sqrt{1+y^2}})\,dy$$

$$= -\iint_D \left\{ \frac{\partial}{\partial x}(3x+e^{y\sqrt{1+y^2}}) - \frac{\partial}{\partial y}(xe^{x^2}+2y) \right\} dx\,dy$$

$$= -\iint_D (3-2)\,dx\,dy$$

$$= -\text{area}(D) = -\frac{1}{2}\times 2 \times 5 = -5$$

$$\therefore \int_{C_1+C_2} = -5 + \int_{C_3} = -5 + \int_2^4 te^{t^2}dt$$

$$= -5 + \left[\frac{1}{2}e^{t^2}\right]_2^4 = \underline{\underline{\frac{1}{2}e^{16} - \frac{1}{2}e^{4} - 5}}$$

**Question 7 (b)** [5 marks]

Find the **exact value** of the surface integral

$$\iint_S (x\mathbf{i} + y\mathbf{j}) \cdot \, d\mathbf{S},$$

where $S$ is the portion of the paraboloid $z = x^2+y^2$ lying below the plane $z = 4$ and oriented with upward pointing normal vectors.

| **Answer 7(b)** | $-16\pi$ |
|---|---|

*(Show your working below and on the next page.)*

$S: \vec{r}(u,v) = u\vec{i} + v\vec{j} + (u^2+v^2)\vec{k}, \quad 0 \le u^2+v^2 \le 4$

$\vec{r}_u = \vec{i} + 0\vec{j} + 2u\vec{k}$

$\vec{r}_v = 0\vec{i} + \vec{j} + 2v\vec{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k}$$

$\vec{r}_u \times \vec{r}_v \cdot \vec{k} = 1 > 0 \Rightarrow \vec{r}_u \times \vec{r}_v$ points upward.

$$\iint_S (x\vec{i} + y\vec{j}) \cdot d\vec{S} = \iint_{0 \le u^2+v^2 \le 4} (u\vec{i} + v\vec{j}) \cdot (-2u\vec{i} - 2v\vec{j} + \vec{k})\, du\,dv$$

$$= \iint_{0 \le u^2+v^2 \le 4} (-2u^2 - 2v^2)\, du\,dv$$

*(More working space for Question 7(b))*

$$= \int_0^{2\pi} \int_0^2 -2r^2 \, r \, dr \, d\theta$$

$$= 2\pi \left[ -\frac{1}{2} r^4 \right]_0^2$$

$$= \underline{\underline{-16\pi}}$$

**Question 8 (a)** [5 marks]

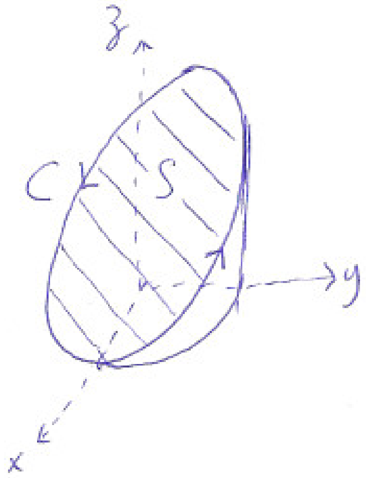
Use Stokes' Theorem to find the **exact value** of the line integral

$$\oint_C \left(-y\,dx + x^2\,dy + z^3\,dz\right),$$

where $C$ is the curve of intersection of the plane $x + z = 3$ and the cylinder $x^2 + y^2 = 4$, oriented counterclockwise as seen from above.

| | |
|---|---|
| **Answer 8(a)** | $4\pi$ |

*(Show your working below and on the next page.)*




Let $S = \{x+z=3\} \cap \{x^2+y^2=4\}$

$\therefore S : \vec{r}(u,v) = u\vec{i} + v\vec{j} + (3-u)\vec{k}$

$0 \le u^2+v^2 \le 4$

$\vec{r}_u = \vec{i} + 0\vec{j} - \vec{k}$

$\vec{r}_v = 0\vec{i} + \vec{j} + 0\vec{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \vec{i} + \vec{k}$$

$\vec{r}_u \times \vec{r}_v \cdot \vec{k} = 1 > 0 \Rightarrow \vec{r}_u \times \vec{r}_v$ points upwards

$\therefore$ using $\vec{r}_u \times \vec{r}_v$ as the orientation of $S$ is compatible to the orientation of $C$ in Stoke's Theorem.

*(More working space for Question 8(a))*

$$\operatorname{curl}\left(-y\vec{i}+x^2\vec{j}+z^3\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x^2 & z^3 \end{vmatrix} = (2x+1)\vec{k}$$

Stoke's Theorem $\Rightarrow \oint_C -y\,dx + x^2\,dy + z^3\,dz$

$$= \iint_{0 \le u^2+v^2 \le 4} (2u+1)\,du\,dv$$

$$= \int_0^2 \int_0^{2\pi} (2r\cos\theta+1)\,d\theta\, r\,dr$$

$$= 2\pi\left[\frac{1}{2}r^2\right]_0^2$$

$$= \underline{\underline{4\pi}}$$

## Question 8 (b) [5 marks]

Use the divergence theorem to evaluate $\iint_S \mathbf{F} \bullet d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = -y^2 z\mathbf{i} + y^2\mathbf{j} + x^3y^3\mathbf{k}$$

and $S$ is the surface of the rectangular region bounded by the three coordinate planes and the planes $x = -1,\ y = 2,\ z = -3$. The orientation of $S$ is given by the outer normal vector.

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| Answer 8(b) | 12 |
|---|---|

*(Show your working below and on the next page.)*

$\operatorname{div} \vec{F} = 2y$

orientation of $S$ is compatible in the Divergence Theorem.

Divergence Theorem $\Rightarrow \iint_S \vec{F} \cdot d\vec{S}$

$$= \int_{-3}^{0} \int_{0}^{2} \int_{-1}^{0} 2y \, dx \, dy \, dz$$

$$= \int_{-3}^{0} \int_{0}^{2} 2y \, dy \, dz$$

$$= \int_{-3}^{0} \left[ y^2 \right]_{y=0}^{y=2} dz$$

$$= \int_{-3}^{0} 4 \, dz = \underline{\underline{12}}$$