2015/2016 SEMESTER 1 MID-TERM TEST

MA1505 MATHEMATICS I

September/October 2015

8:30pm to 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **FOURTEEN (14)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1/10).
4. Use **only 2B pencils** for FORM CC1/10.
5. On FORM CC1/10 (section B for matric numbers starting with A, section C for others), **write** your **matriculation number** and **shade** the corresponding numbered circles **completely**. Your FORM CC1/10 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
6. **Write your full name** in the blank space for module code in section A of FORM CC1/10.
7. Only circles for answers 1 to 10 are to be shaded.
8. For each answer, the circle corresponding to your choice should be **properly** and **completely** shaded. If you change your answer later, you must make sure that the original answer is properly erased.
9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1/10.
11. Submit FORM CC1/10 before you leave the test hall.

Formulae List

1. The **Taylor series** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

2.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

4.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

5.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

6.

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

7. Let $P_n(x)$ be the nth order Taylor polynomial of $f(x)$ at $x = a$. Then

$$f(x) = P_n(x) + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

8. The **projection** of a vector $\mathbf{b}$ onto a vector $\mathbf{a}$, denoted by $\text{proj}_{\mathbf{a}}\mathbf{b}$ is given by

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{(\mathbf{a}\cdot\mathbf{b})}{||\mathbf{a}||^2}\,\mathbf{a}.$$

9. The shortest distance from a point $S\ (x_0,\ y_0,\ z_0)$ to a plane $\Pi\ :\ ax + by + cz = d$, is given by

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$$

1. Let $y = \sin(x^2)$. Find $\frac{dy}{dx}$.

(A) $2x\cos(x^2)$

(B) $-2x\cos(x^2)$

(**C**) $\frac{1}{2}x^2\cos(x^2)$

(**D**) $\frac{1}{3}x^3\cos(x^2)$

(**E**) None of the above

2. Let C denote the curve

$$x^2 + xy - y^2 = 110000.$$

The line $x = 300$ intersects C at two points P_1 and P_2. The tangent line to C at P_1 intersects the tangent line to C at P_2 at the point Q. Find the x-coordinate of Q. Give your answer correct to the nearest integer.

(A) 289

(B) 293

(C) 287

(D) 295

(E) None of the above.

3. A water tank has the shape of an inverted right circular cone with base radius 3 m and height 5 m. Water is being pumped into the tank at a rate of 3 cubic metre per minute. At the moment when the water is 2 m deep the water level is rising at a rate of k m/min. Find the value of k. Give your answer correct to three decimal places.

(Note that the volume V of a right circular cone with base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.)

(A) 0.633

(B) 0.636

(C) 0.663

(D) 0.666

(E) None of the above

4. Let $g(t)$ be a continuous function which satisfies $g(2) = 1$ and $\int_0^2 g(t)\,dt = 1$. Let

$$f(x) = x \int_0^x g(t)\,dt,$$

where $x > 0$. Find the exact value of $f'(2)$.

(A) 1

(B) 2

(C) 3

(D) 4

(E) None of the above

5. Find the area in the first quadrant of the finite region bounded between the curve $y = xe^{-x}$ and the line $y = xe^{-50}$. Give exact value for your answer.

(A) $1 - 1001e^{-50}$

(B) $1 - 1101e^{-50}$

(C) $1 - 1201e^{-50}$

(D) $1 - 1301e^{-50}$

(E) None of the above.

6. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{(n+2)} (x+5)^n .$$

(A) $\frac{1}{2}$

(B) $\frac{1}{4}$

(C) 2

(D) 4

(E) None of the above.

7. Let $\sum_{n=0}^{\infty} a_n (x+1)^n$ denote the Taylor Series of $\frac{1-x}{(x+2)(x+3)}$ at -1. Find the exact value of a_6.

(Suggestion: You may want to use the equation $\frac{1-x}{(x+2)(x+3)} = \frac{3}{x+2} - \frac{4}{x+3}$.)

(A) $\dfrac{85}{32}$

(B) $\dfrac{181}{64}$

(C) $\dfrac{191}{64}$

(D) $\dfrac{95}{32}$

(E) None of the above.

8. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ denote three vectors in three dimensional space. It is known that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 6$. Find the exact value of

$$[(\mathbf{a} + 2\mathbf{b}) \times (3\mathbf{b} + 4\mathbf{c})] \cdot 5\mathbf{c}.$$

(A) 60

(B) 90

(C) 120

(D) 30

(E) None of the above.

9. Two disjoint parallel planes Π_1 and Π_2 are given by:

$$\Pi_1 : \quad 3x + 2y + z \;=\; a, \qquad\qquad \Pi_2 : \quad 3x + 2y + z \;=\; b,$$

where a and b are constants. If the point $(2, 3, 4)$ is equidistant from Π_1 and Π_2, find the exact value of the sum

$$a + b.$$

(A) 32

(B) 64

(C) 16

(D) 128

(E) None of the above.

10. The curve C is given by the vector function

$$\mathbf{r}(t) \;=\; e^t \cos t\, \mathbf{i} \;+\; 2015\, \mathbf{j} \;+\; e^t \sin t\, \mathbf{k},$$

where $0 \le t \le \pi$. Find the length of C.

(A) $e^{2\pi} - 1$

(B) $\sqrt{2}\left(e^{2\pi} - 1\right)$

(C) $e^{\pi} - 1$

(D) $\sqrt{2}\,(e^{\pi} - 1)$

(E) None of the above.

END OF PAPER

Additional blank page for you to do your calculations

National University of Singapore
Department of Mathematics

2015-2016 Semester 1   MA1505 Mathematics I   Mid-Term Test Answers

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Question	1	2	3	4	5	6	7	8	9	10
Answer	A	B	C	C	D	A	D	B	A	D

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$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

1. Let $y = \sin(x^2)$. Find $\frac{dy}{dx}$.

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(C) $\frac{1}{2}x^2\cos(x^2)$

(D) $\frac{1}{3}x^3\cos(x^2)$

(E) None of the above

$$\frac{dy}{dx} = 2x\cos x^2$$

2. Let C denote the curve

$$x^2 + xy - y^2 = 110000.$$

The line $x = 300$ intersects C at two points P_1 and P_2. The tangent line to C at P_1 intersects the tangent line to C at P_2 at the point Q. Find the x-coordinate of Q. Give your answer correct to the nearest integer.

(A) 289

(B) 293

(C) 287

(D) 295

(E) None of the above.

$x=300 \Rightarrow 90000 + 300y - y^2 = 110000 \Rightarrow y = 100$ or 200.

$\frac{d}{dx} \Rightarrow 2x + y + xy' - 2yy' = 0$

at $P_1 = (300, 100) \Rightarrow 600 + 100 + 300y' - 200y' = 0 \Rightarrow y' = -7$

$\therefore$ tangent: $y - 100 = -7(x - 300)$

at $P_2 = (300, 200) \Rightarrow 600 + 200 + 300y' - 400y' = 0 \Rightarrow y' = 8$

$\therefore$ tangent: $y - 200 = 8(x - 300)$

at $Q \Rightarrow -100 = 15(x - 300) \Rightarrow x = \frac{4400}{15} \approx 293.33$

3. A water tank has the shape of an inverted right circular cone with base radius 3 m and height 5 m. Water is being pumped into the tank at a rate of 3 cubic metre per minute. At the moment when the water is 2 m deep the water level is rising at a rate of k m/min. Find the value of k. Give your answer correct to three decimal places.

(Note that the volume V of a right circular cone with base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.)

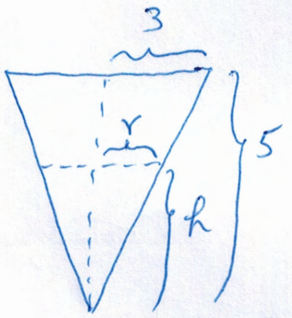
(A) 0.633

(B) 0.636

(C) 0.663

(D) 0.666

(E) None of the above



$\frac{r}{h} = \frac{3}{5} \Rightarrow r = \frac{3}{5}h$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{9}{25}\right)h^3 = \frac{3\pi}{25}h^3$

$\frac{dV}{dt} = \frac{9\pi}{25}h^2 \frac{dh}{dt}$

$\frac{dV}{dt} = 3,\ h = 2 \Rightarrow 3 = \frac{9\pi}{25} \times 4 \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{25}{12\pi} \simeq 0.66315$

4. Let $g(t)$ be a continuous function which satisfies $g(2) = 1$ and $\int_0^2 g(t)\,dt = 1$. Let

$$f(x) = x \int_0^x g(t)\,dt,$$

where $x > 0$. Find the exact value of $f'(2)$.

(A) 1

(B) 2

(C) 3

(D) 4

(E) None of the above

$f'(x) = \left(\int_0^x g(t)dt\right) + x g(x)$

$f'(2) = \left(\int_0^2 g(t)\,dt\right) + 2g(2)$

$= 1 + 2 = \underline{\underline{3}}$

5. Find the area in the first quadrant of the finite region bounded between the curve $y = xe^{-x}$ and the line $y = xe^{-50}$. Give exact value for your answer.

(A) $1 - 1001e^{-50}$

(B) $1 - 1101e^{-50}$

(C) $1 - 1201e^{-50}$

(D) $1 - 1301e^{-50}$

(E) None of the above.

$xe^{-x} = xe^{-50} \Rightarrow x = 0$ or 50

Clearly $xe^{-x} \geq xe^{-50}$ for $0 \leq x \leq 50$

$\therefore$ area $= \int_0^{50} (xe^{-x} - xe^{-50})\,dx$

$= -\int_0^{50} x\,d(e^{-x}) - \left[\frac{1}{2}x^2 e^{-50}\right]_0^{50}$

$= -\left[xe^{-x}\right]_0^{50} + \int_0^{50} e^{-x}\,dx - 1250e^{-50}$

$= -50e^{-50} - \left[e^{-x}\right]_0^{50} - 1250e^{-50}$

$= 1 - 1301e^{-50}$

6. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{(n+2)} (x+5)^n .$$

(A) $\frac{1}{2}$

(B) $\frac{1}{4}$

(C) 2

(D) 4

(E) None of the above.

$$\lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+2}}{n+3}(x+5)^{n+1}}{\frac{(-2)^{n+1}}{n+2}(x+5)^n} \right|$$

$$= \lim_{n\to\infty} 2\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)|x+5|$$

$$= 2|x+5|$$

$$2|x+5| < 1 \Rightarrow |x+5| < \frac{1}{2}$$

7. Let $\sum_{n=0}^{\infty} a_n (x+1)^n$ denote the Taylor Series of $\frac{1-x}{(x+2)(x+3)}$ at -1. Find the exact value of a_6.

(Suggestion: You may want to use the equation $\frac{1-x}{(x+2)(x+3)} = \frac{3}{x+2} - \frac{4}{x+3}$.)

(A) $\frac{85}{32}$

(B) $\frac{181}{64}$

(C) $\frac{191}{64}$

(D) $\frac{95}{32}$

(E) None of the above.

First Solution:

$$f(x) = \frac{3}{x+2} - \frac{4}{x+3}$$

$$f'(x) = (-1)\frac{3}{(x+2)^2} - (-1)\frac{4}{(x+3)^2}$$

$$f''(x) = 2!\,\frac{3}{(x+2)^3} - 2!\,\frac{4}{(x+3)^3}$$

$$f'''(x) = -3!\,\frac{3}{(x+2)^4} - (-1)3!\,\frac{4}{(x+3)^4}$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^n \frac{3(n!)}{(x+2)^{n+1}} - (-1)^n \frac{4(n!)}{(x+3)^{n+1}}$$

$$a_6 = \frac{f^{(6)}(-1)}{n!} = 3 - \frac{4}{2^7} = \frac{95}{32}$$

Second Solution:

$$f(x) = \frac{3}{(x+1)+1} - \frac{4}{(x+1)+2}$$

$$= \frac{3}{1-[-(x+1)]} - \frac{2}{1-\left[-\left(\frac{x+1}{2}\right)\right]}$$

$$= \sum_{n=0}^{\infty} (-1)^n (3)(x+1)^n - \sum_{n=0}^{\infty} (-1)^n (2)\left(\frac{x+1}{2}\right)^n$$

$$a_6 = 3 - 2\left(\frac{1}{2}\right)^6$$

$$= \frac{95}{32}$$

8. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ denote three vectors in three dimensional space. It is known that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 6$. Find the exact value of

$$[(\mathbf{a} + 2\mathbf{b}) \times (3\mathbf{b} + 4\mathbf{c})] \cdot 5\mathbf{c}.$$

(A) 60

(B) 90

(C) 120

(D) 30

(E) None of the above.

$[(\vec{a}+2\vec{b})\times(3\vec{b}+4\vec{c})]\cdot 5\vec{c}$

$=[\vec{a}\times 3\vec{b} + \vec{a}\times 4\vec{c} + 2\vec{b}\times 3\vec{b} + 2\vec{b}\times 4\vec{c}]\cdot 5\vec{c}$

$=(\vec{a}\times 3\vec{b})\cdot 5\vec{c}$ $\quad$ ($\because \vec{a}\times 4\vec{c} \perp 5\vec{c}$, $2\vec{b}\times 4\vec{c} \perp 5\vec{c}$ and $2\vec{b}\times 3\vec{b} = 0$)

$= 15(\vec{a}\times\vec{b})\cdot\vec{c}$

$= \underline{\underline{90}}$

9. Two disjoint parallel planes Π_1 and Π_2 are given by:

$$\Pi_1 : \quad 3x + 2y + z = a, \qquad \Pi_2 : \quad 3x + 2y + z = b,$$

where a and b are constants. If the point $(2, 3, 4)$ is equidistant from Π_1 and Π_2, find the exact value of the sum

$$a + b.$$

(A) 32

(B) 64

(C) 16

(D) 128

(E) None of the above.

$$\frac{|3(2)+2(3)+4-a|}{\sqrt{3^2+2^2+1^2}} = \frac{|3(2)+2(3)+4-b|}{\sqrt{3^2+2^2+1^2}}$$

$\Rightarrow 16 - a = 16 - b$ or $16 - a = -(16 - b)$

$\Rightarrow a = b$ or $a + b = 32$

$\because \Pi_1$ and Π_2 are disjoint

$\therefore a \neq b$

$\therefore a + b = 32$

10. The curve C is given by the vector function

$$\mathbf{r}(t) = e^t \cos t\, \mathbf{i} + 2015\, \mathbf{j} + e^t \sin t\, \mathbf{k},$$

where $0 \leq t \leq \pi$. Find the length of C.

(A) $e^{2\pi} - 1$

(B) $\sqrt{2}\left(e^{2\pi} - 1\right)$

(C) $e^{\pi} - 1$

(D) $\sqrt{2}\,(e^{\pi} - 1)$

(E) None of the above.

END OF PAPER

$\vec{r}'(t) = (e^t \cos t - e^t \sin t)\vec{i} + (e^t \sin t + e^t \cos t)\vec{k}$

$\|\vec{r}'(t)\|^2 = 2e^{2t}$

length $= \int_0^{\pi} \|\vec{r}'(t)\| dt$

$= \sqrt{2} \int_0^{\pi} e^t \, dt$

$= \sqrt{2}\,(e^{\pi} - 1)$

Additional blank page for you to do your calculations