2014/2015 SEMESTER 1 MID-TERM TEST

MA1505 MATHEMATICS I

September/October 2014

8:30pm to 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **FOURTEEN (14)** printed pages.

2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.

3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1/10).

4. Use **only 2B pencils** for FORM CC1/10.

5. On FORM CC1/10 (section B for matric numbers starting with A, section C for others), **write** your **matriculation number** and **shade** the corresponding numbered circles **completely**. Your FORM CC1/10 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.

6. **Write your full name** in the blank space for module code in section A of FORM CC1/10.

7. Only circles for answers 1 to 10 are to be shaded.

8. For each answer, the circle corresponding to your choice should be **properly** and **completely** shaded. If you change your answer later, you must make sure that the original answer is properly erased.

9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.

10. **Do not fold** FORM CC1/10.

11. Submit FORM CC1/10 before you leave the test hall.

Formulae List

1. The **Taylor series** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

2.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

4.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

5.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

6.

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

7. Let $P_n(x)$ be the nth order Taylor polynomial of $f(x)$ at $x = a$. Then

$$f(x) = P_n(x) + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

8. The **projection** of a vector $\mathbf{b}$ onto a vector $\mathbf{a}$, denoted by $\text{proj}_{\mathbf{a}}\mathbf{b}$ is given by

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{(\mathbf{a} \cdot \mathbf{b})}{||\mathbf{a}||^2}\,\mathbf{a}.$$

9. The shortest distance from a point $S\ (x_0,\ y_0,\ z_0)$ to a plane $\Pi\ :\ ax + by + cz = d$, is given by

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

1. Find the slope of the tangent line to the parametric curve

$$x = \cos^3 t, \quad y = \sin^3 t$$

at the point corresponding to $t = \frac{\pi}{4}$.

(A) $\frac{1}{3}$

(B) $-\frac{1}{3}$

(**C**) 1

(**D**) -1

(**E**) None of the above

2. At a point on the curve

$$\sqrt{x} \; + \; \sqrt{y} \; = \; c,$$

where c is a (positive) constant, the tangent has $x-$intercept $(p,\ 0)$ and $y-$intercept $(0,\ q)$. Find the value of $p+q$.

(A) $2\sqrt{c}$

(B) $2c$

(C) c^2

(D) $4\sqrt{c}$

(E) None of the above.

3. For each value of c between 0 and 2, the line $x = c$ intersects the curves $y^2 = 8x$ and $y = x^2$ at the points A and B respectively. Find the maximum possible distance between A and B. Give your answer correct to four decimal places.

(A) 1.8899

(B) 1.8989

(C) 1.9898

(D) 1.9988

(E) None of the above

4. Let

$$f(x) = \int_1^x \ln t dt + \int_x^e (1 - \ln t)\, dt,$$

where $1 < x < e$.

Find $f'(x)$.

(A) $2 \ln x - 1$

(B) $\ln x - 2$

(C) $2 \ln x + 1$

(D) $\ln x + 2$

(E) None of the above

5. Find the value of $\int_0^{\pi} \left(\sqrt{\sin^3 x}\right)(|\cos x|)\, dx$.

(A) $\frac{3}{5}$

(B) $\frac{4}{5}$

(C) $\frac{1}{5}$

(D) $\frac{2}{5}$

(E) None of the above.

6. Find the area enclosed by the closed curve

$$y^2 = 4x^2 - 4x^4$$

with $x \geq 0$.

(A) $\frac{2}{3}$

(B) $\frac{4}{3}$

(C) $\frac{1}{6}$

(D) $\frac{1}{3}$

(E) None of the above.

7. Given the following power series

$$1 + \frac{1}{4}(x-8)^2 + \frac{1}{16}(x-8)^4 + \cdots + \left(\frac{x-8}{2}\right)^{2n} + \cdots ,$$

find all the values of x for which the series converges, expressing the answer as an interval.

(A) $(6,\ 10)$

(B) $(4,\ 12)$

(C) $(0,\ 16)$

(D) $(0,\ 8)$

(E) None of the above.

8. You want to solve the equation

$$e^{-x} = \tan^{-1} x.$$

You know that there is a solution between 0 and 1. To find an approximate value of this solution, you replace e^{-x} by its Taylor polynomial of order 2 at $x = 0$, replace $\tan^{-1} x$ by its Taylor polynomial of order 1 at $x = 0$. After simplifying the resulting expression, you arrive at a quadratic equation. You find a solution to this quadratic equation correct to three decimal places and use it as an approximate answer to your original equation. What is this approximate answer?

(A) 0.613

(B) 0.606

(C) 0.591

(D) 0.586

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9. The three planes Π_1, Π_2 and Π are given by:

$$\Pi_1 : \quad x - y + 4z = 0, \qquad \Pi_2 : \quad 2x - 5y - z = 3,$$

$$\Pi : \quad ax + by + cz = 2,$$

where a, b and c are constants.

The line L is the intersection of Π_1 and Π_2. If Π passes through the point $P(1, 1, 8)$ and is perpendicular to L, find the value of the sum

$$a + b + c.$$

(A) 18

(B) 27

(C) 9

(D) 12

(E) None of the above.

10. The curve C is given by the vector function

$$\mathbf{r}(t) \;=\; (2t^2)\mathbf{i} \;+\; (2t^4)\mathbf{j} \;+\; (\ln t)\mathbf{k},$$

where $1 \leq t \leq e^{2014}$. Find the length of C.

(A) $2\left(e^{4028} + e^{8056} - 2\right) + 2014$

(B) $2\left(e^{8056} - 1\right) + 2014$

(C) $4e^{2014} + 8e^{6042} + e^{-2014}$

(D) $4e^{4028} + 8e^{6042} + e^{-2014}$

(E) None of the above.

END OF PAPER

Additional blank page for you to do your calculations

National University of Singapore
Department of Mathematics

2014-2015 Semester 1 MA1505 Mathematics I Mid-Term Test Answers

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Question	1	2	3	4	5	6	7	8	9	10
Answer	D	C	A	A	B	B	A	D	C	B

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1. Find the slope of the tangent line to the parametric curve

$$x = \cos^3 t, \quad y = \sin^3 t$$

at the point corresponding to $t = \frac{\pi}{4}$.

(A) $\frac{1}{3}$

(B) $-\frac{1}{3}$

(C) 1

(D) -1

(E) None of the above

$$\frac{dx}{dt} = 3\cos^2 t\,(-\sin t)$$

$$\frac{dy}{dt} = 3\sin^2 t\cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\tan t$$

at $t = \frac{\pi}{4}$, $\frac{dy}{dx} = -\tan\frac{\pi}{4}$

$$= -1$$

2. At a point on the curve

$$\sqrt{x} + \sqrt{y} = c,$$

where c is a (positive) constant, the tangent has $x-$intercept $(p,\ 0)$ and $y-$intercept $(0,\ q)$. Find the value of $p+q$.

(A) $2\sqrt{c}$

(B) $2c$

(C) c^2

(D) $4\sqrt{c}$

(E) None of the above.

By implicit differentiation,

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\sqrt{\frac{y}{x}}.$$

Let $P(x_0, y_0)$ be a point on the curve. An equation of the tangent at P is:

$$\frac{y-y_0}{x-x_0} = -\sqrt{\frac{y_0}{x_0}} \qquad (*)$$

Setting $y=0$ in $(*)$ gives

$$\frac{-y_0}{p-x_0} = -\sqrt{\frac{y_0}{x_0}} \implies p = x_0 + \sqrt{x_0 y_0}.$$

Similarly, setting $x=0$ in $(*)$ gives

$$\frac{q-y_0}{-x_0} = -\sqrt{\frac{y_0}{x_0}} \implies q = y_0 + \sqrt{x_0 y_0}.$$

Thus,

$$p+q = x_0 + y_0 + 2\sqrt{x_0 y_0} = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = c^2.$$

3. For each value of c between 0 and 2, the line $x = c$ intersects the curves $y^2 = 8x$ and $y = x^2$ at the points A and B respectively. Find the maximum possible distance between A and B. Give your answer correct to four decimal places.

(A) 1.8899

(B) 1.8989

(C) 1.9898

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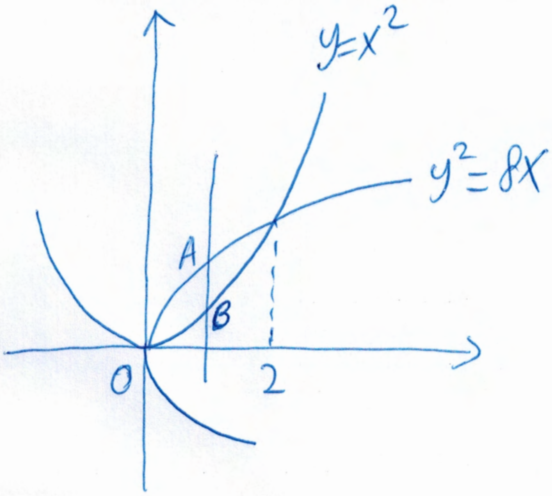
(E) None of the above



For $0 \le x \le 2$,

Length of $AB = f(x) = \sqrt{8x} - x^2 = 2\sqrt{2}\sqrt{x} - x^2$

$$f'(x) = \frac{\sqrt{2}}{\sqrt{x}} - 2x = \frac{\sqrt{2} - 2x^{3/2}}{\sqrt{x}}$$

$$f'(x) = 0 \Rightarrow x^{3/2} = \frac{\sqrt{2}}{2} \Rightarrow x = \left(\tfrac{1}{2}\right)^{1/3}$$

$$f\{(\tfrac{1}{2})^{1/3}\} = 2\sqrt{2}(\tfrac{1}{2})^{1/6} - (\tfrac{1}{2})^{2/3} \simeq 1.88988$$

$$f(0) = f(2) = 0$$

$\therefore$ max. $f \simeq 1.8899$

4. Let

$$f(x) = \int_1^x \ln t dt + \int_x^e (1 - \ln t)\, dt,$$

where $1 < x < e$.

Find $f'(x)$.

(A) $2 \ln x - 1$

(B) $\ln x - 2$

(C) $2 \ln x + 1$

(D) $\ln x + 2$

(E) None of the above

$f(x) = \int_1^x \ln t\, dt - \int_e^x (1 - \ln t)\, dt$

$f'(x) = \ln x - (1 - \ln x)$

$= 2 \ln x - 1$

5. Find the value of $\int_0^{\pi} \left(\sqrt{\sin^3 x}\right)(|\cos x|)\,dx$.

(A) $\frac{3}{5}$

(B) $\frac{4}{5}$

(C) $\frac{1}{5}$

(D) $\frac{2}{5}$

(E) None of the above.

$$\int_0^{\pi} \sqrt{\sin^3 x}\,|\cos x|\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\sin^3 x}\cos x\,dx - \int_{\frac{\pi}{2}}^{\pi} \sqrt{\sin^3 x}\cos x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} x\,d(\sin x) - \int_{\frac{\pi}{2}}^{\pi} \sin^{\frac{3}{2}} x\,d(\sin x)$$

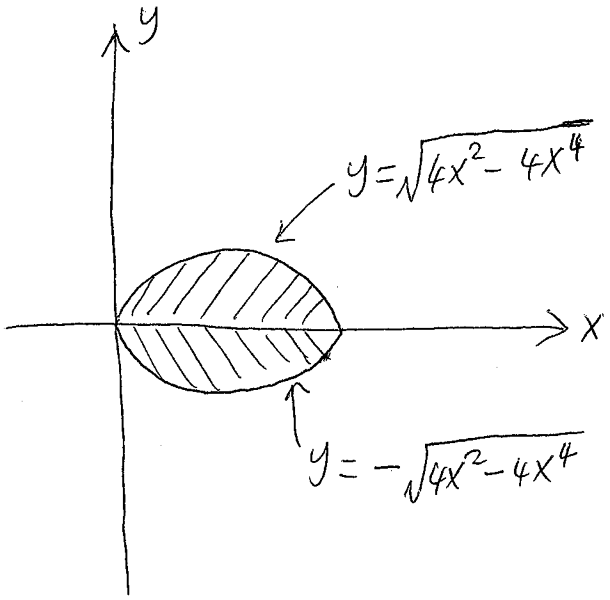
$$= \frac{2}{5}\sin^{\frac{5}{2}} x \Big|_0^{\frac{\pi}{2}} - \frac{2}{5}\sin^{\frac{5}{2}} x \Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{4}{5}$$

6. Find the area enclosed by the closed curve

$$y^2 = 4x^2 - 4x^4$$

with $x \geq 0$.



(A) $\frac{2}{3}$

(B) $\frac{4}{3}$

(C) $\frac{1}{6}$

(D) $\frac{1}{3}$

(E) None of the above.

Area enclosed = 2 × area of upper half

$$= 2\int_0^1 \sqrt{4x^2-4x^4}\,dx$$

$$= 2\int_0^1 2x\sqrt{1-x^2}\,dx$$

$$= -2\int_0^1 (1-x^2)^{1/2}\,d(1-x^2)$$

$$= -2\left(\tfrac{2}{3}\right)(1-x^2)^{3/2}\Big|_0^1$$

$$= \frac{4}{3}$$

7. Given the following power series

$$1+\frac{1}{4}(x-8)^2+\frac{1}{16}(x-8)^4+\cdots+\left(\frac{x-8}{2}\right)^{2n}+\cdots,$$

find all the values of x for which the series converges, expressing the answer as an interval.

(A) (6, 10)

(B) (4, 12)

(C) (0, 16)

(D) (0, 8)

(E) None of the above.

Observe that this is a geometric series with common ratio $\frac{1}{4}(x-8)^2$.

For convergence, we must have

$$\left|\frac{1}{4}(x-8)^2\right|<1$$

$$\therefore |x-8|^2<4$$

$$|x-8|<2$$

$$\therefore x\in(6,10)$$

8. You want to solve the equation

$$e^{-x} = \tan^{-1} x.$$

You know that there is a solution between 0 and 1. To find an approximate value of this solution, you replace e^{-x} by its Taylor polynomial of order 2 at $x = 0$, replace $\tan^{-1} x$ by its Taylor polynomial of order 1 at $x = 0$. After simplifying the resulting expression, you arrive at a quadratic equation. You find a solution to this quadratic equation correct to three decimal places and use it as an approximate answer to your original equation. What is this approximate answer?

(A) 0.613

(B) 0.606

(C) 0.591

(D) 0.586

(E) None of the above.

$1 - x + \frac{x^2}{2} = x$

$x^2 - 4x + 2 = 0$

$x = 2 \pm \sqrt{2}$

$\because 0 < x < 1, \therefore x = 2 - \sqrt{2} \approx 0.586$

9. The three planes Π_1, Π_2 and Π are given by:

$$\Pi_1 : \quad x - y + 4z = 0, \qquad \Pi_2 : \quad 2x - 5y - z = 3,$$

$$\Pi : \quad ax + by + cz = 2,$$

where a, b and c are constants.

The line L is the intersection of Π_1 and Π_2. If Π passes through the point $P(1,1,8)$ and is perpendicular to L, find the value of the sum

$$a + b + c.$$

(A) 18

(B) 27

(C) 9

(D) 12

(E) None of the above.

Normal vectors to Π_1 and Π_2 are respectively

$$\mathbf{n}_1 = \mathbf{i} - \mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \mathbf{n}_2 = 2\mathbf{i} - 5\mathbf{j} - \mathbf{k}.$$

The vector

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 4 \\ 2 & -5 & -1 \end{vmatrix} = 21\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}$$

is parallel to L. Since Π passes through $P(1,1,8)$ and is perpendicular to L, it follows that an equation of Π is

$$21x + 9y - 3z = 6, \quad \text{i.e.} \quad 7x + 3y - z = 2.$$

Thus, $a + b + c = 7 + 3 - 1 = 9.$

10. The curve C is given by the vector function

$$\mathbf{r}(t) = (2t^2)\mathbf{i} + (2t^4)\mathbf{j} + (\ln t)\mathbf{k},$$

where $1 \le t \le e^{2014}$. Find the length of C.

(A) $2\left(e^{4028} + e^{8056} - 2\right) + 2014$

(B) $2\left(e^{8056} - 1\right) + 2014$

(C) $4e^{2014} + 8e^{6042} + e^{-2014}$

(D) $4e^{4028} + 8e^{6042} + e^{-2014}$

(E) None of the above.

END OF PAPER

First obtain

$$\mathbf{r}'(t) = (4t)\mathbf{i} + (8t^3)\mathbf{j} + \left(\frac{1}{t}\right)\mathbf{k},$$

which gives

$$\|\mathbf{r}'(t)\| = \sqrt{(4t)^2 + (8t^3)^2 + \left(\frac{1}{t}\right)^2} = \sqrt{16t^2 + 64t^6 + \frac{1}{t^2}}$$
$$= 8t^3 + \frac{1}{t}.$$

Thus, the length of C is given by

$$\int_1^{e^{2014}} \|\mathbf{r}'(t)\|\,dt = \int_1^{e^{2014}} \left(8t^3 + \frac{1}{t}\right) dt = \left[2t^4 + \ln t\right]_1^{e^{2014}}$$
$$= 2\left(e^{8056} - 1\right) + 2014.$$

Additional blank page for you to do your calculations