Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2013-2014

MA1505 MATHEMATICS I

November 2013 Time allowed: 2 hours

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INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** Do not write your name anywhere in this booklet. This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question. The marks for each question are indicated at the beginning of the question. The maximum possible total score for this examination paper is 80 marks.
4. This is a **closed book (with authorized material)** examination. Students are only allowed to bring into the examination hall **ONE** piece A4 size help-sheet which must be handwritten and can be written on both sides.
5. Candidates may use any non-programmable and non-graphing calculators. However, they should lay out systematically the various steps in the calculations.

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Question	1	2	3	4	5	6	7	8
(a)								
(b)								

Question 1 (a) [5 marks] (Multiple Choice Question)

Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series for the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \cos \frac{x}{2} \quad \text{if } -\pi < x < \pi$$

and $f(x) = f(x + 2\pi)$ for all $x \in \mathbb{R}$.
Find the **exact value** of a_0.

(A) $\frac{1}{\pi}$ (B) $\frac{1}{2\pi}$ (C) $\frac{2}{\pi}$ (D) $\frac{2}{3\pi}$

Answer 1(a)	C or $\frac{2}{\pi}$

(Show your working below and on the next page.)

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos \frac{x}{2} \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \cos \frac{x}{2} \, dx$$

$$= \frac{2}{\pi} \sin \frac{x}{2} \Big|_0^{\pi}$$

$$= \underline{\underline{\frac{2}{\pi}}}$$

Question 1 (b) [5 marks]

Let $f(x)$ be a function defined by

$$f(x) = \sin|x| + 5\sin(2013x) \quad \text{if } -\pi < x < \pi ,$$

and $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$.
Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series for $f(x)$.
(i) Find the **exact value** of a_2.
(ii) Find the **exact value** of

$$\sum_{n=1}^{\infty} b_n.$$

Answer 1(b)(i)	$-\frac{4}{3\pi}$	**Answer 1(b)(ii)**	5

(Show your working below and on the next page.)

Observe that $f(x) = \underbrace{\sin|x|}_{\text{even part}} + \underbrace{5\sin 2013x}_{\text{odd part}}$

(i) $a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin|x| \cos 2x \, dx$

$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos 2x \, dx$

$= \frac{2}{\pi}\int_0^{\pi} (2\cos^2 x - 1)\sin x \, dx$

$= -\frac{2}{\pi}\int_0^{\pi} (2\cos^2 x - 1)\, d(\cos x)$

$= \left[-\frac{4}{3\pi}\cos^3 x + \frac{2}{\pi}\cos x\right]_0^{\pi} = -\frac{4}{3\pi}$

(ii) $b_n = \begin{cases} 5 & \text{if } n = 2013 \\ 0 & \text{if } n \neq 2013 \end{cases}$

$\therefore \sum_{n=1}^{\infty} b_n = 5$

Question 2 (a) [5 marks]

Let $f(x)$ be a function defined on the open interval $(0,1)$ by

$$f(x)=\begin{cases} x & \text{if } 0<x<\frac{1}{2}\,; \\ 0 & \text{if } \frac{1}{2}<x<1\,. \end{cases}$$

Find the first two non-zero terms of the sine Fourier half range expansion for $f(x)$. Give **exact values** for your answer.

Answer 2(a)	$\frac{2}{\pi^2}\sin\pi x+\frac{1}{2\pi}\sin 2\pi x$

(Show your working below and on the next page.)

$$b_n = 2\int_0^1 f(x)\sin n\pi x\,dx$$

$$= 2\int_0^{\frac{1}{2}} x\sin n\pi x\,dx = -\frac{2}{n\pi}\int_0^{\frac{1}{2}} x\,d(\cos n\pi x)$$

$$= -\frac{2}{n\pi}\left\{ x\cos n\pi x\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}}\cos n\pi x\,dx\right\}$$

$$= -\frac{2}{n\pi}\left\{\frac{1}{2}\cos\frac{n\pi}{2} - \frac{1}{n\pi}\sin n\pi x\Big|_0^{\frac{1}{2}}\right\}$$

$$= -\frac{2}{n\pi}\left\{\frac{1}{2}\cos\frac{n\pi}{2} - \frac{1}{n\pi}\sin\frac{n\pi}{2}\right\}$$

$$\therefore b_1 = \frac{2}{\pi^2}$$

$$b_2 = \frac{1}{2\pi}$$

$$\therefore f(x) \sim \frac{2}{\pi^2}\sin\pi x + \frac{1}{2\pi}\sin 2\pi x + \cdots$$

Question 2 (b) [5 marks]

Let f be the function given by

$$f(x, y, z) = x^2y + z^2.$$

(i) Find the gradient of f at the point $(1, 2, 3)$. Give **exact values** for your answer.
(ii)Find the directional derivative of f at the point $(1, 2, 3)$ in the direction of the vector that joins the point $(1, 2, 3)$ to the point $(-1, 1, 5)$. Give **exact value** for your answer.

Answer 2(b)(i)	$4\vec{i}+\vec{j}+6\vec{k}$ or $(4,1,6)$ or $\begin{pmatrix}4\\1\\6\end{pmatrix}$	**Answer 2(b)(ii)**	1

(Show your working below and on the next page.)

(i) $\nabla f = 2xy\,\vec{i} + x^2\vec{j} + 2z\,\vec{k}$

$\nabla f(1,2,3) = \underline{\underline{4\vec{i} + \vec{j} + 6\vec{k}}}$

(ii) $\vec{u} = \dfrac{(-1,1,5)-(1,2,3)}{\|(-1,1,5)-(1,2,3)\|} = \dfrac{(-2,-1,2)}{3} = -\frac{2}{3}\vec{i} - \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$

$D_{\vec{u}} f(1,2,3) = \nabla f(1,2,3)\cdot\vec{u}$

$= -\frac{8}{3} - \frac{1}{3} + 4 = \underline{\underline{1}}$

Question 3 (a) [5 marks]

The upper cone C has a parametric representation given by

$$C : \quad \mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (1 + 5\sqrt{u^2+v^2})\mathbf{k}.$$

Find an equation of the tangent plane to C at the point $P(-3,-4,26)$.
Give your answer in the form

$$ax + by + cz = 1.$$

Answer 3(a)	$3x+4y+z=1$

(Show your working below and on the next page.)

$$\vec{r}_u = \vec{i} + 0\vec{j} + \frac{5u}{\sqrt{u^2+v^2}}\vec{k}$$

$$\vec{r}_v = 0\vec{i} + \vec{j} + \frac{5v}{\sqrt{u^2+v^2}}\vec{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{5u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{5v}{\sqrt{u^2+v^2}} \end{vmatrix} = -\frac{5u}{\sqrt{u^2+v^2}}\vec{i} - \frac{5v}{\sqrt{u^2+v^2}}\vec{j} + \vec{k}$$

$$\vec{r}_u \times \vec{r}_v\,(-3,-4,26) = 3\vec{i} + 4\vec{j} + \vec{k}$$

$$\therefore\ 3x + 4y + z = -9 - 16 + 26 = 1$$

$$\underline{\underline{3x+4y+z=1}}$$

Question 3 (b) [5 marks]

Find the absolute maximum value of the function $f(x, y, z) = xyz$ subject to the constraint $3x^2 + 2y^2 + 6z^2 = 18$. Give **exact value** for your answer.

Answer 3(b)	$\sqrt{6}$

(Show your working below and on the next page.)

Let $F(x,y,z,\lambda) = xyz - \lambda(3x^2+2y^2+6z^2-18)$

$F_x = 0 \Rightarrow yz - 6\lambda x = 0$ ---- ①

$F_y = 0 \Rightarrow xz - 4\lambda y = 0$ ----- ②

$F_z = 0 \Rightarrow xy - 12\lambda z = 0$ ----- ③

$F_\lambda = 0 \Rightarrow 3x^2+2y^2+6z^2-18=0$ --- ④

① $\Rightarrow 6\lambda x^2 = xyz$; ② $\Rightarrow 4\lambda y^2 = xyz$; ③ $\Rightarrow 12\lambda z^2 = xyz$

$\therefore 6\lambda x^2 = 4\lambda y^2 = 12\lambda z^2$

$\therefore 3x^2 = 2y^2 = 6z^2$

$\therefore$ ④ $\Rightarrow 3x^2+3x^2+3x^2=18 \Rightarrow x = \pm\sqrt{2}$

$\therefore y = \pm\sqrt{3}$

$z = \pm 1$

$\therefore xyz = \pm\sqrt{6}$

$\max f = \underline{\underline{\sqrt{6}}}$

Question 4 (a) [5 marks]

Find the **exact value** of the double integral

$$\int\int_D e^{y^2}\,dxdy,$$

where D is the finite domain bounded by the graph of $y = |x|$ and the line $y = 1$.

Answer 4(a)	$e - 1$

(Show your working below and on the next page.)

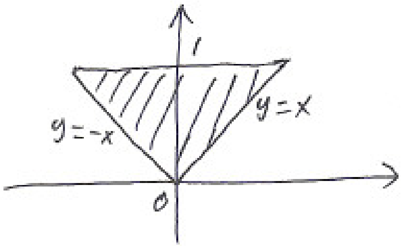
$$\iint_D e^{y^2}dxdy$$

$$= \int_0^1 \int_{-y}^{y} e^{y^2}dx\,dy$$

$$= \int_0^1 2ye^{y^2}dy$$

$$= e^{y^2}\Big|_0^1$$

$$= \underline{\underline{e-1}}$$



Question 4 (b) [5 marks]

The region S is the portion of the sphere

$$x^2 + y^2 + z^2 = 14z$$

that lies within the paraboloid

$$x^2 + y^2 = 2z.$$

Find the **exact** surface area of S.

Answer 4(b)	28π

(Show your working below and on the next page.)

$$2z + z^2 = 14z \Rightarrow z = 0 \text{ or } 12$$

$$z = 0 \Rightarrow x^2+y^2 = 0 \Rightarrow x = 0 \text{ and } y = 0$$

$$z = 12 \Rightarrow x^2+y^2 = 24$$

$\therefore$ the paraboloid and the sphere intersect at $(0,0,0)$ and $C: x^2+y^2=24,\ z=12$

Next $S: z^2 - 14z + (x^2+y^2) = 0$

$$\Rightarrow z = \frac{14 \pm \sqrt{196-4(x^2+y^2)}}{2} = 7 \pm \sqrt{49-x^2-y^2}$$

$\because z = 12$, $\therefore z = 7 + \sqrt{49-x^2-y^2}$

area of $S = \displaystyle\iint_{x^2+y^2\le 24} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dx\,dy = \iint_{x^2+y^2\le 24} \frac{7}{\sqrt{49-x^2-y^2}}\, dx\,dy$

$$= \int_0^{2\pi}\int_0^{\sqrt{24}} \frac{7}{\sqrt{49-r^2}}\, r\,dr\,d\theta = \underline{\underline{28\pi}}$$

Question 5 (a) [5 marks] (Multiple Choice Question)

Find the **exact value** of the iterated integral

$$\int_0^1 \left[\int_0^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\, dx \right] dy.$$

(A) π (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Answer 5(a)	B or $\frac{\pi}{6}$

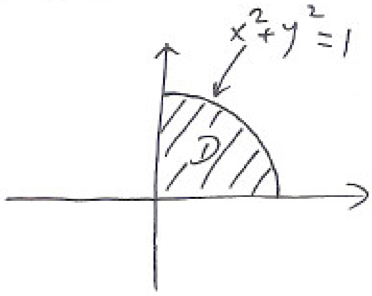
(Show your working below and on the next page.)

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\, dx\, dy$$

$$= \iint_D \sqrt{x^2+y^2}\, dx\, dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 r\,(r\,dr\,d\theta)$$

$$= \underline{\underline{\frac{\pi}{6}}}$$

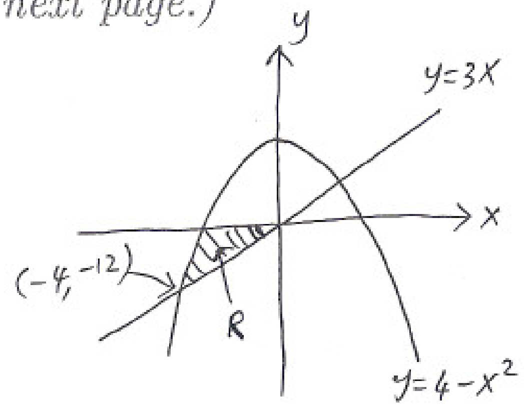


Question 5 (b) [5 marks]

Let R denote the finite region in the **third quadrant** of the xy-plane bounded by the parabola $y = 4 - x^2$, the x-axis and the line $y = 3x$. The solid D has base R, sides perpendicular to the xy-plane, and top lying on the surface $z = xy$. Find the **exact value** of the volume of D.

Answer 5(b)	144

(Show your working below and on the next page.)



Volume of D

$= \iint_R xy \, dA$

$= \int_{-12}^{0} \int_{-\sqrt{4-y}}^{\frac{1}{3}y} xy \, dx \, dy$

$= \int_{-12}^{0} \left[\frac{1}{2}x^2 y\right]_{x=-\sqrt{4-y}}^{x=\frac{1}{3}y} dy$

$= \int_{-12}^{0} \left\{\frac{1}{18}y^3 - \frac{1}{2}(4-y)y\right\} dy$

$= \left[\frac{1}{72}y^4 - y^2 + \frac{1}{6}y^3\right]_{-12}^{0}$

$= \underline{\underline{144}}$

Question 6 (a) [5 marks]

Let $g(x)$ denote a continuously differentiable function of one variable x which satisfies $g(0) = 2$. Assume that the value of the line integral

$$I = \int_C xy^2 dx + yg(x) dy$$

is independent of the path C which joins $(0,0)$ to $(2,3)$.

(i) Find an **exact** formula for the function $g(x)$.

(ii) Find the **exact value** of the line integral I.

Answer 6(a)(i)	x^2+2	Answer 6(a)(ii)	27

(Show your working below and on the next page.)

(i) $\frac{\partial}{\partial y}(xy^2) = \frac{\partial}{\partial x}(yg(x)) \Rightarrow 2xy = y g'(x)$

$\Rightarrow g'(x) = 2x$

$\Rightarrow g(x) = x^2 + C$

$g(0) = 2 \Rightarrow C = 2$

$\therefore g(x) = \underline{\underline{x^2+2}}$

(ii) $f_x = xy^2 \Rightarrow f = \frac{1}{2}x^2y^2 + k(y)$

$\therefore f_y = x^2y + k'(y)$

$\therefore k'(y) = 2y$

$\therefore k = y^2 + C_1 \Rightarrow f = \frac{1}{2}x^2y^2 + y^2 + C_1$

$I = f(2,3) - f(0,0) = \underline{\underline{27}}$

Question 6 (b) [5 marks]

The curve C is the upper semi-circle in the xy-plane with equation

$$C : \quad x^2 + y^2 = 1, \quad y \geq 0.$$

If

$$\mathbf{F}(x, y) = 16y\,\mathbf{i}$$

is a force field that moves an object along C in the counterclockwise direction from $(1, 0)$ to $(-1, 0)$, find the **exact value** of the work done $\int_C \mathbf{F} \bullet d\mathbf{r}$.

Answer 6(b)	-8π

(Show your working below and on the next page.)

$$C : \vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}, \quad 0 \leq t \leq \pi$$

$$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} -16 \sin^2 t \, dt$$

$$= -8 \int_0^{\pi} (1 - \cos 2t)\, dt$$

$$= \underline{\underline{-8\pi}}.$$

Question 7 (a) [5 marks]

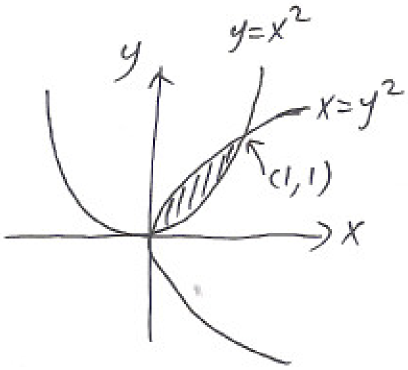
Let D be the finite region in the first quadrant of the xy-plane bounded by the curves $y = x^2$ and $x = y^2$. Let C be the boundary of D with ~~a~~ counterclockwise orientation. Find the **exact value** of the line integral

$$\oint_C \left(2xy - x^2 - e^{x^3}\right) dx \;+\; (x + y^2 + \cos^{128} y)\, dy \;.$$

Answer 7(a)	$\frac{1}{30}$

(Show your working below and on the next page.)

$$\oint_C = \iint_D \left\{ \frac{\partial}{\partial x}(x+y^2+\cos^{128} y) - \frac{\partial}{\partial y}(2xy - x^2 - e^{x^3}) \right\} dA$$



$$= \iint_D (1-2x)\, dA$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (1-2x)\, dx\, dy$$

$$= \int_0^1 (\sqrt{y} - y - y^2 + y^4)\, dy$$

$$= \underline{\underline{\frac{1}{30}}}$$

Question 7 (b) [5 marks]

Find the **exact value** of the surface integral

$$\iint_S z\,dS$$

where S is the closed surface bounded laterally by S_1: the cylinder $x^2+y^2=9$; bounded below by S_2: the horizontal plane $z=1$ and bounded above by S_3: the horizontal plane $z=3$.

Answer 7(b)	60π

(Show your working below and on the next page.)

$S_1: \vec{r}(u,v) = 3\cos u\,\vec{i} + 3\sin u\,\vec{j} + v\vec{k}$, $0 \le u \le 2\pi$, $1 \le v \le 3$.

$\vec{r}_u = -3\sin u\,\vec{i} + 3\cos u\,\vec{j} + 0\vec{k}$

$\vec{r}_v = 0\vec{i} + 0\vec{j} + \vec{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ -3\sin u & 3\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\cos u\,\vec{i} + 3\sin u\,\vec{j}$$

$\|\vec{r}_u \times \vec{r}_v\| = 3$

$$\iint_{S_1} z\,dS = \int_1^3 \int_0^{2\pi} 3v\,du\,dv = 24\pi$$

$$\iint_{S_2} z\,dS = \iint_{S_2} 1\,dS = 9\pi$$

$$\iint_{S_3} z\,dS = \iint_{S_3} 3\,dS = 27\pi$$

$$\iint_S z\,dS = 24\pi + 9\pi + 27\pi = \underline{\underline{60\pi}}$$

Question 8 (a) [5 marks]

Evaluate $\iint_S \mathbf{F} \bullet d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k},$$

and S is the part of the paraboloid $z = 1 - x^2 - y^2$ with $z \geq 0$. The orientation of S is given by the upward normal vector.

Answer 8(a)	2π

(Show your working below and on the next page.)

$S: \vec{r}(u,v) = u\vec{i} + v\vec{j} + (1-u^2-v^2)\vec{k}$

$\vec{r}_u = \vec{i} + 0\vec{j} - 2u\vec{k}$

$\vec{r}_v = 0\vec{i} + \vec{j} - 2v\vec{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = 2u\vec{i} + 2v\vec{j} + \vec{k}$$

$(\vec{r}_u \times \vec{r}_v) \cdot \vec{k} = 1 = +ve \Rightarrow \vec{r}_u \times \vec{r}_v$ points upward

Projection of S onto uv plane $= D = \{(u,v) : 0 \leq u^2+v^2 \leq 1\}$

$$\iint_S \vec{F} \cdot dS = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

$$= \iint_D \{2u^2 + 2v^2 + 2(1-u^2-v^2)\}\, du\, dv$$

$$= 2(\text{area of } D)$$

$$= \underline{\underline{2\pi}}$$

Question 8 (b) [5 marks]

Let $\mathbf{F}$ be the vector field given by

$$\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}.$$

Let C denote the circle with centre at $(0,0,0)$ and radius equals to 2 that lies on the plane $x + y + z = 0$. The orientation of C is counterclockwise when viewed from above.

(i) Find curl($\mathbf{F}$).

(ii) Use Stokes' Theorem to find the **exact value** of the line integral

$$\oint_C \mathbf{F} \bullet d\mathbf{r} .$$

Answer 8(b)(i)	$-\vec{i}-\vec{j}-\vec{k}$ or $(-1,-1,-1)$ or $\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$	**Answer 8(b)(ii)**	$-4\sqrt{3}\pi$

(Show your working below and on the next page.)

(i) $\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \underline{\underline{-\vec{i} - \vec{j} - \vec{k}}}$

(ii) Plane has normal unit vector $\vec{n} = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}$

$\because$ $\vec{n}$ points upwards, $\therefore$ it is compatible to orientation of C.

$\therefore \oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \vec{n}\, dS = -\sqrt{3}$ area of S

$= -\sqrt{3}(4\pi)$

$= \underline{\underline{-4\sqrt{3}\pi}}$