Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2012-2013

MA1505 MATHEMATICS I

November 2012 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
(a)								
(b)								

Question 1 (a) [5 marks] (Multiple Choice Question)

Let $f(x)$ be a function defined by

$$f(x) = 1505 + 1506x + 1507x^2 + 1508x^3 \quad \text{if } -\pi < x < \pi\,,$$

and $f(x + 2\pi) = f(x)$.

Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series for $f(x)$.

Find the **exact value** of

$$a_0 + \sum_{n=1}^{\infty} a_n.$$

(A) 1505 (B) 1506 (C) 1507 (D) 1508

Answer 1(a)	(A) or 1505

(Show your working below and on the next page.)

$\because f$ is continuous at $x=0$

$\therefore a_0 + \sum_{n=1}^{\infty} a_n = f(0) = \underline{\underline{1505}}$

Question 1 (b) [5 marks]

Find the first two non-zero terms of the Fourier series of the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^2 \quad \text{if } -1 \le x < 1$$

and $f(x) = f(x+2)$ for all $x \in \mathbb{R}$. Give **exact values** for your answer.

Answer 1(b)	$\frac{1}{3} - \frac{4}{\pi^2}\cos \pi x$

(Show your working below and on the next page.)

f is even $\Rightarrow b_n = 0$ for $n = 1, 2, 3, \ldots$

$$a_0 = \int_0^1 x^2 dx = \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{3}$$

$$\begin{aligned} a_1 &= 2\int_0^1 x^2 \cos \pi x \, dx \\ &= \frac{2}{\pi}\int_0^1 x^2 \, d(\sin \pi x) \\ &= \frac{2}{\pi}\left\{ \left[x^2 \sin \pi x\right]_0^1 - \int_0^1 2x \sin \pi x \, dx \right\} \\ &= \frac{4}{\pi^2}\int_0^1 x \, d(\cos \pi x) \\ &= \frac{4}{\pi^2}\left\{ \left[x \cos \pi x\right]_0^1 - \int_0^1 \cos \pi x \, dx \right\} \\ &= -\frac{4}{\pi^2} \end{aligned}$$

$$\therefore f(x) \sim \underline{\underline{\frac{1}{3} - \frac{4}{\pi^2}\cos \pi x}} + \cdots$$

Question 2 (a) [5 marks]

Let $f(x)$ be a function defined on the open interval $(0, \pi)$ by

$$f(x) = \cos x \quad .$$

Find the first two non-zero terms of the sine Fourier half range expansion for $f(x)$. Give **exact values** for your answer.

Answer 2(a)	$\frac{8}{3\pi} \sin 2x + \frac{16}{15\pi} \sin 4x$

(Show your working below and on the next page.)

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \, dx$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{-1}{2\pi}\left[\cos 2x\right]_0^{\pi} = 0$$

$$n \geq 2 \Rightarrow b_n = \frac{1}{\pi}\int_0^{\pi}\{\sin(n+1)x + \sin(n-1)x\}\,dx$$

$$= \frac{1}{\pi}\left\{\left[-\frac{\cos(n+1)x}{n+1}\right]_0^{\pi} + \left[-\frac{\cos(n-1)x}{n-1}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n+1}((-1)^n + 1) + \frac{1}{n-1}((-1)^n + 1)\right\}$$

$$b_2 = \frac{8}{3\pi}, \quad b_3 = 0, \quad b_4 = \frac{16}{15\pi}, \quad \ldots$$

$$f(x) \sim \frac{8}{3\pi}\sin 2x + \frac{16}{15\pi}\sin 4x + \cdots$$

Question 2 (b) [5 marks]

Let f be the function given by

$$f(x, y, z) = x^2 z + y^2.$$

Find the directional derivative of f at the point $(1, 0, 1)$ in the direction of the unit vector $\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$. Give **exact values** for your answer.

Answer 2(b)	$\sqrt{3}$

(Show your working below and on the next page.)

$$\nabla f(x,y,z) = (2xz,\ 2y,\ x^2)$$

$$\nabla f(1,0,1) = (2,\ 0,\ 1)$$

$$\vec{u} = \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$D_{\vec{u}} f(1,0,1) = \nabla f(1,0,1) \cdot \vec{u}$$

$$= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$= \underline{\underline{\sqrt{3}}}$$

Question 3 (a) [5 marks]

Suppose $f(x, y)$ has continuous partial derivatives of all orders. If

$$\nabla f = \left(xy^2 + kx^2y + x^3\right)\mathbf{i} + \left(x^3 + x^2y + y^2\right)\mathbf{j},$$

find the value of the constant k.

Answer 3(a)	3

(Show your working below and on the next page.)

$$\frac{\partial}{\partial y}(xy^2 + kx^2y + x^3) = \frac{\partial}{\partial x}(x^3 + x^2y + y^2)$$

$$\Rightarrow \quad 2xy + kx^2 = 3x^2 + 2xy$$

$$\Rightarrow \quad k = 3.$$

Question 3 (b) [5 marks]

A company manufactures a product P using three types of inputs A,B and C. When x units of A, y units of B and z units of C are used, the company can make $18x^2yz$ units of P. The company can buy A, B and C at \$24, \$18, and \$12 per unit respectively. What is the maximum number of P that it can produce if it has a budget of \$144?

Answer 3(b)	972

(Show your working below and on the next page.)

$f(x,y,z) = 18x^2yz = \max!$

$g(x,y,z) = 24x + 18y + 12z - 144 = 0$

$F(x,y,z) = 18x^2yz - \lambda(24x + 18y + 12z - 144)$

$F_x = 0 \Rightarrow 36xyz - 24\lambda = 0 \Rightarrow \lambda = \frac{3}{2}xyz \cdots (1)$

$F_y = 0 \Rightarrow 18x^2z - 18\lambda = 0 \Rightarrow \lambda = x^2z \cdots (2)$

$F_z = 0 \Rightarrow 18x^2y - 12\lambda = 0 \Rightarrow \lambda = \frac{3}{2}x^2y \cdots (3)$

(1) & (2) $\Rightarrow x = \frac{3}{2}y$

(2) & (3) $\Rightarrow z = \frac{3}{2}y$

$\therefore 24(\frac{3}{2}y) + 18y + 12(\frac{3}{2}y) = 144$

$72y = 144 \Rightarrow y = 2$

$\therefore x = 3,\ y = 2,\ z = 3$

$\max f = 18(3^2)(2)(3) = \underline{\underline{972}}$

Question 4 (a) [5 marks]

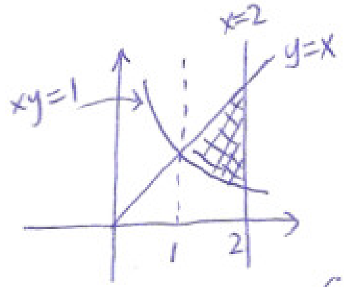
Find the **exact value** of the double integral

$$\int\int_D \left(\frac{x^2}{y^2}\right) dxdy,$$

where D is the finite domain bounded by the curve $xy = 1$ and the two lines: $x = 2$, $y = x$.

Answer 4(a)	$\frac{9}{4}$

(Show your working below and on the next page.)



$D: \frac{1}{x} \leq y \leq x, \quad 1 \leq x \leq 2$

$$\iint_D \frac{x^2}{y^2} dx\,dy = \int_1^2 \int_{\frac{1}{x}}^{x} \frac{x^2}{y^2} dy\,dx$$

$$= \int_1^2 \left[-\frac{x^2}{y}\right]_{y=\frac{1}{x}}^{y=x} dx$$

$$= \int_1^2 (-x + x^3)\,dx$$

$$= \left[-\frac{1}{2}x^2 + \frac{1}{4}x^4\right]_1^2$$

$$= 2 + \frac{1}{4} = \underline{\underline{\frac{9}{4}}}$$

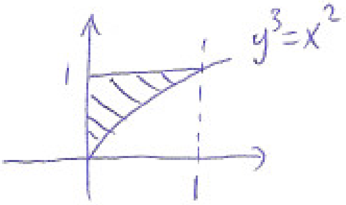
Question 4 (b) [5 marks]

Find the **exact value** of the iterated integral

$$\int_0^1 \int_{x^{2/3}}^1 x\cos(y^4)\,dy\,dx.$$

Answer 4(b)	$\frac{1}{8}\sin 1$

(Show your working below and on the next page.)



$$\int_0^1 \int_{x^{2/3}}^1 x\cos y^4\,dy\,dx$$

$$= \int_0^1 \int_0^{y^{3/2}} x\cos y^4\,dx\,dy$$

$$= \int_0^1 \cos y^4 \left[\tfrac{1}{2}x^2\right]_{x=0}^{x=y^{3/2}} dy$$

$$= \int_0^1 \tfrac{1}{2}y^3\cos y^4\,dy$$

$$= \tfrac{1}{8}\sin y^4 \Big|_0^1$$

$$= \underline{\underline{\tfrac{1}{8}\sin 1}}$$

Question 5 (a) [5 marks]

The solid D lies on the xy-plane within the circular cylinder

$$x^2 + y^2 = 1$$

and is bounded above by the plane $z = 2x + 3$. Find the **exact** volume of D.

Answer 5(a)	3π

(Show your working below and on the next page.)

$$\text{Vol} = \iint_{0 \le x^2+y^2 \le 1} (2x+3)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 (2r\cos\theta + 3)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{2}{3}r^3\cos\theta + \frac{3}{2}r^2\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{2}{3}\cos\theta + \frac{3}{2}\right) d\theta$$

$$= \underline{\underline{3\pi}}$$

Question 5 (b) [5 marks]

Find the **exact value** of the surface area of that portion of the sphere $x^2 + y^2 + z^2 = 3$ that lies above the plane $z = 1$.

Answer 5(b)	$(6-2\sqrt{3})\pi$

(Show your working below and on the next page.)

$z=1 \Rightarrow x^2+y^2+1=3 \Rightarrow x^2+y^2=2$

$z=\sqrt{3-x^2-y^2} \Rightarrow z_x = \dfrac{-x}{\sqrt{3-x^2-y^2}}, \quad z_y = \dfrac{-y}{\sqrt{3-x^2-y^2}}$

$\sqrt{1+z_x^2+z_y^2} = \sqrt{\dfrac{3}{3-x^2-y^2}}$

$$\text{Surface area} = \iint_{0\le x^2+y^2\le 2} \sqrt{\frac{3}{3-x^2-y^2}}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{\frac{3}{3-r^2}}\, r\,dr\,d\theta$$

$$= 2\pi \int_0^{\sqrt{2}} \left(-\frac{\sqrt{3}}{2}\right)(3-r^2)^{-\frac{1}{2}}\,d(3-r^2)$$

$$= 2\pi \left[-\sqrt{3}\,(3-r^2)^{\frac{1}{2}}\right]_{r=0}^{r=\sqrt{2}}$$

$$= 2\pi(-\sqrt{3}+3)$$

$$= \underline{\underline{(6-2\sqrt{3})\pi}}$$

Question 6 (a) [5 marks] (Multiple Choice Question)

Consider the following vector field:

$$\mathbf{F}(x, y, z) = (2x, 2z, y).$$

Let C be the straight line from $(0, 0, 0)$ to $(1, 2, 3)$. Find the **exact value** of the line integral:

$$\int_C \mathbf{F} \bullet d\mathbf{r}.$$

(A) 5 (B) 8 (C) 10 (D) 13

Answer 6(a)	(C) or 10

(Show your working below and on the next page.)

$C : \vec{r}(t) = (t, 2t, 3t), \quad 0 \le t \le 1$

$\vec{F}(\vec{r}(t)) = (2t, 6t, 2t)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (2t, 6t, 2t) \cdot (1, 2, 3)\, dt$$

$$= \int_0^1 (2t + 12t + 6t)\, dt$$

$$= \int_0^1 20t\, dt$$

$$= 10t^2 \Big|_0^1 = \underline{\underline{10}}$$

Question 6 (b) [5 marks]

Let C be the helix parametrised by

$$\mathbf{r}(t) = (3\cos t, 3\sin t, 4t) \quad \text{for } 0 \leq t \leq 4\pi,$$

and let $f(x, y, z) = x^2 + \frac{1}{16}z$. Find the **exact value** of the line integral

$$\int_C f\,ds.$$

Answer 6(b)	$90\pi + 10\pi^2$

(Show your working below and on the next page.)

$$\vec{r}'(t) = (-3\sin t, 3\cos t, 4)$$

$$\|\vec{r}'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$$

$$\int_C f\,ds = \int_0^{4\pi} \left(9\cos^2 t + \frac{1}{4}t\right) 5\,dt$$

$$= \int_0^{4\pi} \left\{\frac{45}{2}(1+\cos 2t) + \frac{5}{4}t\right\} dt$$

$$= \left[\frac{45}{2}t + \frac{45}{4}\sin 2t + \frac{5}{8}t^2\right]_0^{4\pi}$$

$$= \underline{\underline{90\pi + 10\pi^2}}$$

Question 7 (a) [5 marks]

Let C be the circle centred at $(3,4)$ and of radius 2. Find the **exact value** of the line integral

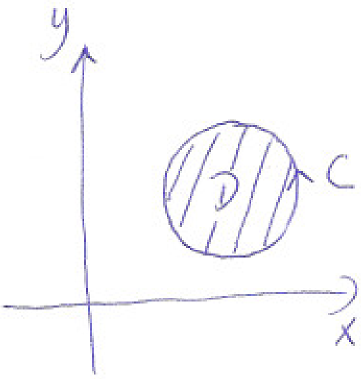
$$\oint_C \left(\frac{1}{y} - e^{2x}\right) dx + \left(7x - \frac{x}{y^2}\right) dy,$$

where C is oriented anticlockwise.

Answer 7(a)	28π

(Show your working below and on the next page.)

Green's Theorem



$$\Rightarrow \oint_C \left(\frac{1}{y} - e^{2x}\right) dx + \left(7x - \frac{x}{y^2}\right) dy$$

$$= \iint_D \left\{\frac{\partial}{\partial x}\left(7x - \frac{x}{y^2}\right) - \frac{\partial}{\partial y}\left(\frac{1}{y} - e^{2x}\right)\right\} dxdy$$

$$= \iint_D \left(7 - \frac{1}{y^2} + \frac{1}{y^2}\right) dxdy$$

$$= 7\,(\text{area of } D)$$

$$= 7\,(\pi 2^2)$$

$$= \underline{\underline{28\pi}}$$

Question 7 (b) [5 marks]

Find the **exact value** of the surface integral

$$\iint_S \mathbf{F} \bullet d\mathbf{S},$$

where

$$\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

and S is the portion of the plane

$$2x + y + z = 2$$

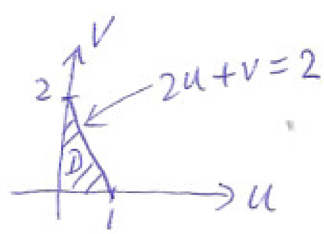
in the first octant. The orientation of S is given by the upward normal vector.

Answer 7(b)	$\frac{7}{3}$

(Show your working below and on the next page.)

$S : \vec{r}(u,v) = u\vec{i} + v\vec{j} + (2-2u-v)\vec{k}$

with $(u,v) \in D$.



$\vec{r}_u = \vec{i} + 0\vec{j} - 2\vec{k}$

$\vec{r}_v = 0\vec{i} + \vec{j} - \vec{k}$

(More working space for Question 7(b))

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = 2\vec{i} + \vec{j} + \vec{k}$$

$\vec{r}_u \times \vec{r}_v \cdot \vec{k} = 1 > 0 \Rightarrow \vec{r}_u \times \vec{r}_v$ points upwards

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(v\vec{i} + (2-2u-v)\vec{j} + u\vec{k}\right) \cdot (2\vec{i} + \vec{j} + \vec{k})\, dA$$

$$= \iint_D (2v + 2 - 2u - v + u)\, dA$$

$$= \iint_D (v - u + 2)\, dA$$

$$= \int_0^1 \int_0^{2-2u} (v - u + 2)\, dv\, du$$

$$= \int_0^1 \left[\frac{1}{2}v^2 - uv + 2v\right]_{v=0}^{v=2-2u} du$$

$$= \int_0^1 (4u^2 - 10u + 6)\, du$$

$$= \left[\frac{4}{3}u^3 - 5u^2 + 6u\right]_0^1$$

$$= \frac{7}{3}$$

Question 8 (a) [5 marks]

Let S be the upper hemisphere with equation

$$S \quad : \quad z = \sqrt{1 - x^2 - y^2}.$$

If

$$\mathbf{F}(x, y, z) = z^2\mathbf{i} - 2x\mathbf{j} + y^3\mathbf{k},$$

find the **exact value** of the surface integral $\iint_S \text{curl } \mathbf{F} \bullet d\mathbf{S}$, where the orientation of S is given by the outer normal vector.

Answer 8(a)	-2π

(Show your working below and on the next page.)

$C = \partial S : \vec{r}(t) = \cos t \,\vec{i} + \sin t \,\vec{j} + 0\vec{k}$

$0 \leq t \leq 2\pi$

orientation of C is compatible to the orientation of S in Stoke's Theorem.

Stoke's Theorem

$\Rightarrow \iint_S \text{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$

$= \int_0^{2\pi} (0^2 \vec{i} - 2\cos t \,\vec{j} + \sin^3 t \,\vec{k}) \cdot (-\sin t \,\vec{i} + \cos t \,\vec{j} + 0\vec{k})\, dt$

$= \int_0^{2\pi} -2\cos^2 t \, dt$

$= \int_0^{2\pi} (-1 - \cos 2t)\, dt = \underline{\underline{-2\pi}}$

Question 8 (b) [5 marks]

Let W be the cube bounded by the three coordinate planes $x = 0$, $y = 0$, $z = 0$ and the three planes $x = 2$, $y = 2$, $z = 2$. Let S be the surface consisting of five sides of W, excluding the side where $z = 0$. Orient S with outward pointing normal vector. Let $\mathbf{F}$ be the vector field given by

$$\mathbf{F}(x,y,z) = \left(10x - 3xy + \cos y^2\right)\mathbf{i} + \left(z^2 e^x + \cos x^2\right)\mathbf{j} + (3zy - 1)\,\mathbf{k}.$$

Find the **exact value** of the surface integral $\iint_S \mathbf{F} \bullet d\mathbf{S}$.

Answer 8(b)	76

(Show your working below and on the next page.)

Let S_1 be the side of W with $z=0$.

$S_1: \vec{r}(u,v) = u\vec{i} + v\vec{j} + 0\vec{k}$, $0 \le u \le 2, 0 \le v \le 2$

$\vec{r}_u \times \vec{r}_v = \vec{i} \times \vec{j} = \vec{k}$

Orient S_1 with $\vec{k}$.

Note that $\vec{k}$ points inward to W on S_1.

$\therefore \partial W$ with positive orientation $= S - S_1$

$\operatorname{div} \vec{F} = 10x - 3y + 3y = 10$

Divergence Theorem $\Rightarrow \iint_{S-S_1} \vec{F} \cdot d\vec{S} = \iiint_W (\operatorname{div} \vec{F})\, dV$

$= 10(\text{vol. of } W) = 80$

$\therefore \iint_S \vec{F} \cdot d\vec{S} = 80 + \iint_{S_1} \vec{F} \cdot d\vec{S}$

$= 80 + \iint_{\substack{0 \le u \le 2 \\ 0 \le v \le 2}} (-1)\, du\, dv = 80 - 4 = \underline{\underline{76}}$