2012/2013 SEMESTER 1 MID-TERM TEST

MA1505 MATHEMATICS I

2 October 2012

8:30pm to 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **FOURTEEN (14)** printed pages.
2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.
3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1/10).
4. Use **only 2B pencils** for FORM CC1/10.
5. On FORM CC1/10 (section B for matric numbers starting with A, section C for others), **write** your **matriculation number** and **shade** the corresponding numbered circles **completely**. Your FORM CC1/10 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.
6. **Write your full name** in the blank space for module code in section A of FORM CC1/10.
7. Only circles for answers 1 to 10 are to be shaded.
8. For each answer, the circle corresponding to your choice should be **properly** and **completely** shaded. If you change your answer later, you must make sure that the original answer is properly erased.
9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.
10. **Do not fold** FORM CC1/10.
11. Submit FORM CC1/10 before you leave the test hall.

Formulae List

1. The **Taylor series** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

2.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

4.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

5.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

6.

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

7. Let $P_n(x)$ be the nth order Taylor polynomial of $f(x)$ at $x = a$. Then

$$f(x) = P_n(x) + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

8. The **projection** of a vector $\mathbf{b}$ onto a vector $\mathbf{a}$, denoted by $\text{proj}_{\mathbf{a}}\mathbf{b}$ is given by

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{(\mathbf{a} \cdot \mathbf{b})}{||\mathbf{a}||^2}\ \mathbf{a}.$$

9. The shortest distance from a point $S\ (x_0,\ y_0,\ z_0)$ to a plane $\Pi \quad : \quad ax + by + cz = d$, is given by

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

1. Find the equation of the tangent line to the curve $x^{\frac{1}{4}} + y^{\frac{1}{4}} = 4$ at the point $(16, 16)$.

(A) $y = -x + 32$

(B) $y = x$

(C) $y = -\frac{1}{2}x + 24$

(D) $y = 2x - 16$

(E) None of the above

2. Let a be a positive constant. The equation $r = a\theta$ represents the spiral of Archimedes in polar co-ordinates. Find the slope of the tangent to this spiral at the point $\theta = \frac{\pi}{2}$.

(A) $\dfrac{\pi}{4}$

(B) a

(C) $-\dfrac{4}{\pi}$

(D) $-\dfrac{2}{\pi}$

(E) None of the above

3. A light shines from the top of a lamp post 15 m high. A ball is dropped from the same height from a point 10 m away from the light. It is known that the ball falls a distance $s = 5t^2$ m in t seconds. Find the speed of the shadow of the ball on the ground 1 second later.

(A) 60 m/s

(B) 75 m/s

(C) 45 m/s

(D) 90 m/s

(E) None of the above

4. Evaluate

$$\int_1^{3^{2012}} \frac{(1+\sqrt{x})^2}{\sqrt{x}}\,dx.$$

(A) $2\left(3^{1006} + 3^{2011} + 3^{3017}\right) - \frac{14}{3}$

(B) $2\left(3^{1006} + 3^{2012} + 3^{3017}\right) - \frac{14}{3}$

(C) $2\left(3^{1007} + 3^{2012} + 3^{3017}\right) - \frac{14}{3}$

(D) $2\left(3^{1006} + 3^{2012} + 3^{3018}\right) - \frac{14}{3}$

(E) None of the above

5. Find the area of the region bounded by the curves $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq (2^{1505})\pi$.

(A) $\dfrac{2^{1508}}{\sqrt{2}}$

(B) $\dfrac{2^{1507}}{\sqrt{2}}$

(C) $\dfrac{2^{1506}}{\sqrt{2}}$

(D) $\dfrac{2^{1505}}{\sqrt{2}}$

(E) None of the above.

6. What is the Taylor series of the function $f(x) = \frac{1}{(1-x)^2}$ around the point $x = 0$?

(A) $1 + \sum_{n=2}^{\infty}(n+1)x^n$

(B) $1 + \sum_{n=0}^{\infty}(n+1)x^{n+2}$

(C) $1 + \sum_{n=2}^{\infty} nx^{n-1}$

(D) $1 + \sum_{n=1}^{\infty} nx^n$

(E) None of the above

7. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n.$$

(A) $\frac{3}{2}$

(B) 2

(C) $\frac{1}{2}$

(D) 4

(E) None of the above.

8. Let P denote the plane passing through the points $(1, 2, 3)$, $(2, 3, 4)$ and $(2, 1, 6)$. Let a denote the distance from the point $(0, 0, 0)$ to P. Let b denote the distance from the point $(1, 1, 1)$ to P. Find the value of the product ab.

(A) 1

(B) 1.5

(C) 2

(D) 2.5

(E) None of the above

9. The triangle OAB has vertices $O(0,0,0)$, $A(2,-1,1)$ and $B(2,2,4)$.

   Find the exact area of the triangle OAB.

   (A) $3\sqrt{3}$

   (B) $9\sqrt{3}$

   (C) $9\sqrt{2}$

   (D) $3\sqrt{2}$

   (E) None of the above.

10. Compute the arclength of the curve

$$\mathbf{r}(t) = (2t)\,\mathbf{i} + \left(t^2\right)\mathbf{j} + \left(\frac{t^3}{3}\right)\mathbf{k},$$

from $t = 0$ to $t = 1$.

(A) $\frac{7}{3}$

(B) $\frac{8}{3}$

(C) $\frac{5}{3}$

(D) 2

(E) None of the above

END OF PAPER

Additional blank page for you to do your calculations

National University of Singapore
Department of Mathematics

2012-2013 Semester 1   MA1505 Mathematics I   Mid-Term Test Answers

Question	1	2	3	4	5	6	7	8	9	10
Answer	A	D	A	B	B	C	D	B	A	A

Mid-term Test Solutions

1). A

$$x^{\frac{1}{4}} + y^{\frac{1}{4}} = 4$$

$$\frac{d}{dx} \Rightarrow \frac{1}{4}x^{-\frac{3}{4}} + \frac{1}{4}y^{-\frac{3}{4}}y' = 0$$

$$\Rightarrow y' = -\frac{x^{-\frac{3}{4}}}{y^{-\frac{3}{4}}} = -\left(\frac{y}{x}\right)^{3/4}$$

at $(16, 16)$, $y' = -\left(\frac{16}{16}\right)^{3/4} = -1$

$$\therefore y - 16 = -1(x - 16) = -x + 16$$

$$y = -x + 32$$

2). D

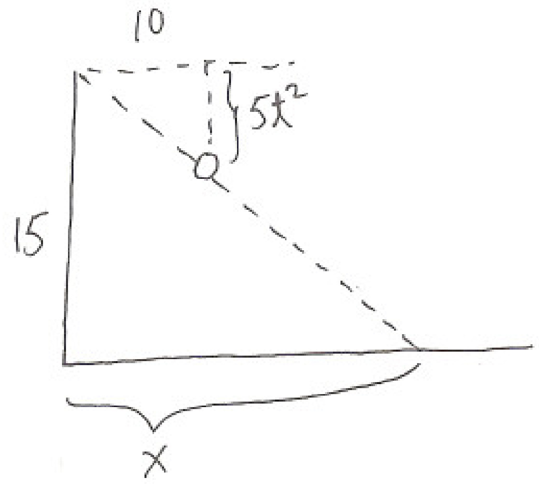
$$r = a\theta \Rightarrow \frac{dr}{d\theta} = a$$

$$y = r\sin\theta \Rightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$x = r\cos\theta \Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\theta = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a}{-a\left(\frac{\pi}{2}\right)} = -\frac{2}{\pi}$$

3). A



$$\frac{x}{10} = \frac{15}{5t^2} \Rightarrow x = \frac{30}{t^2}$$

$$\frac{dx}{dt} = -\frac{60}{t^3}$$

$$t = 1 \Rightarrow \frac{dx}{dt} = -60$$

$\therefore$ speed $= \underline{\underline{60 \text{ m/s}}}$

4). B

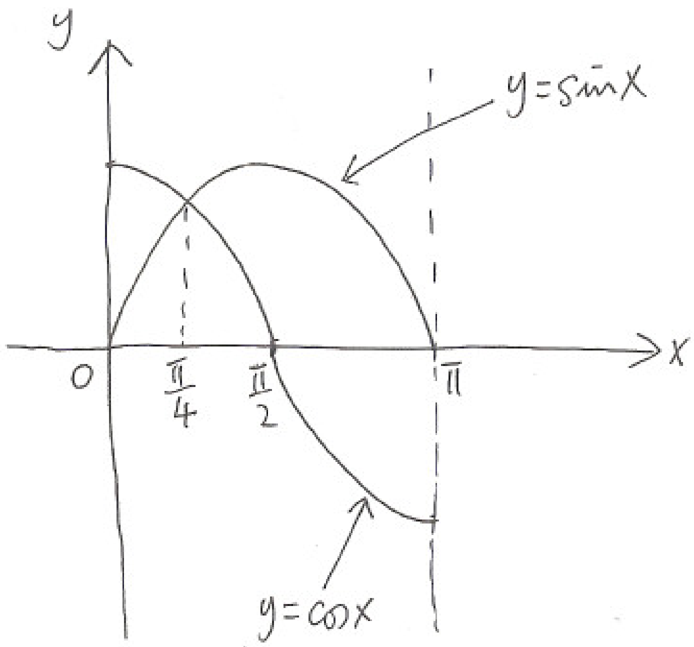
$$\int_1^{3^{2012}} \frac{(1+\sqrt{x})^2}{\sqrt{x}} dx = \int_1^{3^{2012}} \frac{1+2\sqrt{x}+x}{\sqrt{x}} dx$$

$$= \int_1^{3^{2012}} (x^{-\frac{1}{2}} + 2 + x^{\frac{1}{2}}) dx$$

$$= \left[2x^{\frac{1}{2}} + 2x + \frac{2}{3}x^{3/2}\right]_1^{3^{2012}}$$

$$= \underline{\underline{2\left\{3^{1006} + 3^{2012} + 3^{3017}\right\} - \frac{14}{3}}}$$

5). B



area bounded by $y=\sin x$ and $y=\cos x$ for $0 \le x \le \pi$

equals to $\int_0^{\pi} |\sin x - \cos x| \, dx$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi}$$

$$= \frac{2}{\sqrt{2}} - 1 + 1 + \frac{2}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}}$$

By symmetry, area for $0 \le x \le 2^{1505}\pi$

$$= \frac{4}{\sqrt{2}}(2^{1505}) = \frac{2^{1507}}{\sqrt{2}}$$

6). C

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{d}{dx} \Rightarrow \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

7). D

$$\left| \frac{\frac{[(n+1)!]^2}{(2n+2)!} x^{n+1}}{\frac{(n!)^2}{(2n)!} x^n} \right| = \frac{(n+1)^2}{(2n+1)(2n+2)} |x|$$

$$\longrightarrow \frac{1}{4}|x|$$

$$\frac{1}{4}|x| < 1 \Rightarrow |x| < 4$$

8). B

Let $\vec{u} = (1, 2, 3) - (2, 3, 4) = (-1, -1, -1)$

$\vec{v} = (2, 1, 6) - (2, 3, 4) = (0, -2, 2)$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & -1 \\ 0 & -2 & 2 \end{vmatrix} = -4\vec{i} + 2\vec{j} + 2\vec{k}$$

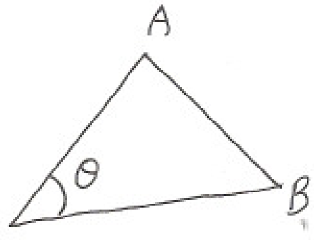
$P: -4x + 2y + 2z = (-4, 2, 2) \cdot (1, 2, 3) = 6$

$\therefore \ 2x - y - z + 3 = 0$

$$a = \frac{|3|}{\sqrt{6}}, \quad b = \frac{|2-1-1+3|}{\sqrt{6}} = \frac{3}{\sqrt{6}}$$

$$\therefore \ ab = \frac{9}{6} = \underline{\underline{1.5}}$$

9). A



$$\text{area} = \frac{1}{2} \|\vec{OA}\| \|\vec{OB}\| \sin\theta$$
$$= \frac{1}{2} \|\vec{OA} \times \vec{OB}\|$$

$$\vec{OA} \times \vec{OB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 2 & 2 & 4 \end{vmatrix} = -6\vec{i} - 6\vec{j} + 6\vec{k}$$

$$\therefore \ \text{area} = \frac{1}{2}(6\sqrt{3}) = \underline{\underline{3\sqrt{3}}}$$

10). A

$$\vec{r}'(t) = 2\vec{i} + 2t\vec{j} + t^2\vec{k}$$

$$\|\vec{r}'(t)\| = \sqrt{4+4t^2+t^4} = t^2+2$$

$$\text{arclength} = \int_0^1 (t^2+2)\,dt$$

$$= \left[\frac{t^3}{3} + 2t\right]_0^1$$

$$= \frac{7}{3}$$