**Matriculation Number:**

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NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2011-2012

# MA1505 MATHEMATICS I

November 2011 Time allowed: 2 hours

## INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

**For official use only. Do not write below this line.**

| Question | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| (a) | | | | | | | | |
| (b) | | | | | | | | |

**Question 1 (a)** [5 marks]

Find the slope of the tangent to the curve

$$y = x + e^x$$

when $x = 1$. Give **exact value** for your answer in terms of $e$.

| **Answer 1(a)** | $1+e$ |
|---|---|

*(Show your working below and on the next page.)*

$$\frac{dy}{dx} = 1 + e^x$$

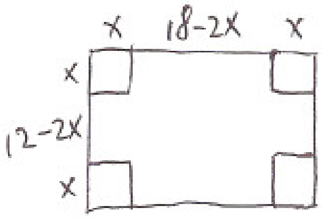
$$\left.\frac{dy}{dx}\right|_{x=1} = \underline{\underline{1+e}}$$

**Question 1 (b)** [5 marks]

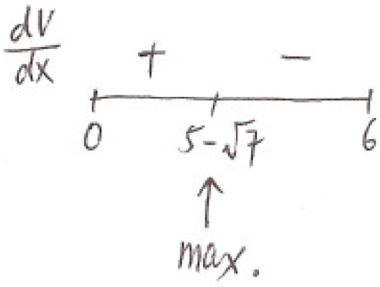
An open box (*without a top cover*) is to be made from a rectangular piece of metal sheet, 12 cm by 18 cm, by cutting out equal squares of side $x$ cm from each of the four corners and folding up the sides (to be perpendicular to the base). Find the **exact value** of $x$ to make the box with the largest possible volume.

| | |
|---|---|
| **Answer 1(b)** | $5-\sqrt{7}$ |

*(Show your working below and on the next page.)*




$$V = x(12-2x)(18-2x)$$
$$= 4x^3 - 60x^2 + 216x, \quad 0 \le x \le 6.$$
$$\frac{dV}{dx} = 12x^2 - 120x + 216$$
$$= 12(x^2 - 10x + 18)$$
$$\frac{dV}{dx} = 0 \Rightarrow x = 5 \pm \sqrt{7}$$
$$= \underline{\underline{5-\sqrt{7}}} \quad (\because 0 \le x \le 6)$$



**Question 2 (a)** [5 marks]

Find the **exact value** of the integral

$$\int_0^{\pi/4} \ln\left(\sqrt{1+\tan x}\right) dx.$$

| Answer 2(a) | $\frac{\pi}{16}\ln 2$ (or $\frac{\pi}{8}\ln\sqrt{2}$) |
|---|---|

*(Show your working below and on the next page.)*

Let $x = \frac{\pi}{4} - t$

$$\int_0^{\pi/4} \ln\left(\sqrt{1+\tan x}\right) dx = \int_{\pi/4}^{0} \ln\left(\sqrt{1+\tan(\tfrac{\pi}{4}-t)}\right)(-dt)$$

$$= \int_0^{\pi/4} \ln\sqrt{1+\frac{\tan\frac{\pi}{4}-\tan t}{1+\tan\frac{\pi}{4}\tan t}}\, dt$$

$$= \int_0^{\pi/4} \ln\sqrt{\frac{2}{1+\tan t}}\, dt = \int_0^{\pi/4}\left(\ln\sqrt{2} - \ln\sqrt{1+\tan t}\right) dt$$

$$2\int_0^{\pi/4} \ln\sqrt{1+\tan x}\, dx = \int_0^{\pi/4} \ln\sqrt{2}\, dt$$

$$= \frac{\pi}{8}\ln 2$$

$$\int_0^{\pi/4} \ln\sqrt{1+\tan x}\, dx = \underline{\underline{\frac{\pi}{16}\ln 2}}$$

**Question 2 (b)** [5 marks]

Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2x+3)^{2n+3}}{2^{n+3}(2n+3)}.$$

| Answer 2(b) | $\frac{\sqrt{2}}{2}$ (or $\frac{1}{\sqrt{2}}$) |
|---|---|

*(Show your working below and on the next page.)*

$$\left| \frac{\dfrac{(2x+3)^{2n+5}}{2^{n+4}(2n+5)}}{\dfrac{(2x+3)^{2n+3}}{2^{n+3}(2n+3)}} \right| = \left| \frac{(2x+3)^2}{2} \cdot \frac{(2n+3)}{(2n+5)} \right|$$

$$\longrightarrow \frac{|2x+3|^2}{2}$$

$$\frac{|2x+3|^2}{2} < 1$$

$$\Leftrightarrow |2x+3| < \sqrt{2}$$

$$\Leftrightarrow \left|x + \frac{3}{2}\right| < \underline{\underline{\frac{\sqrt{2}}{2}}}$$

**Question 3 (a)** [5 marks]

Let $f(x)$ be a function defined by

$$f(x) = x^{2010} + x \quad \text{if} - \pi < x < \pi ,$$

and $f(x + 2\pi) = f(x)$.
Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series for $f(x)$.
(i) Find the **exact value** of $a_0$.
(ii) Find the **exact value** of $b_{2011}$.

| Answer 3(a)(i) | $\frac{\pi^{2010}}{2011}$ | Answer 3(a)(ii) | $\frac{2}{2011}$ |
|---|---|---|---|

*(Show your working below and on the next page.)*

Note that $x^{2010}$ is the even part
$x$ is the odd part

$$\therefore a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^{2010}dx = \frac{1}{\pi}\int_0^{\pi} x^{2010}dx = \underline{\underline{\frac{\pi^{2010}}{2011}}}$$

$$b_{2011} = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin 2011x\,dx = \frac{2}{\pi}\int_0^{\pi} x \sin 2011x\,dx$$

$$= -\frac{2}{2011\pi}\int_0^{\pi} x\, d(\cos 2011x)$$

$$= -\frac{2}{2011\pi}\left\{ x\cos 2011x \Big|_0^{\pi} - \int_0^{\pi} \cos 2011x\,dx \right\}$$

$$= \underline{\underline{\frac{2}{2011}}}$$

**Question 3 (b)** [5 marks]

Let

$$f(x) = \cos x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2},$$

and $f(x+\pi) = f(x)$. Write down the Fourier series expansion for $f(x)$ up to and including the first two non-zero terms. Give **exact values** in terms of $\pi$ in the simplest form for your answer.

| **Answer 3(b)** | $\frac{2}{\pi} + \frac{4}{3\pi}\cos 2x + \cdots$ |
|---|---|

*(Show your working below and on the next page.)*

Note that $\cos x$ is even, $L = \frac{\pi}{2}$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos x\,dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x\,dx$$

$$= \frac{2}{\pi}\sin x\Big|_0^{\pi/2} = \frac{2}{\pi}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos x \cos 2nx\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2}\left[\frac{1}{2}\cos(2n+1)x + \frac{1}{2}\cos(2n-1)x\right]dx$$

$$= \frac{2}{\pi}\left[\frac{\sin(2n+1)x}{2n+1} + \frac{\sin(2n-1)x}{2n-1}\right]_0^{\pi/2}$$

$$a_1 = \frac{2}{\pi}\left[\frac{\sin\frac{3}{2}\pi}{3} + \frac{\sin\frac{\pi}{2}}{1}\right] = \frac{2}{\pi}\left(-\frac{1}{3}+1\right) = \frac{4}{3\pi}$$

$$\therefore f(x) \sim \frac{2}{\pi} + \frac{4}{3\pi}\cos 2x + \cdots$$

**Question 4 (a)** [5 marks]

Find the equation of the plane that passes through the three points

$$(0,0,0), \quad (2,1,1), \quad (1,1,2).$$

Give your answer in the form $ax + by + cz = d$, where $a$, $b$, $c$, $d$ are integers.

| | |
|---|---|
| **Answer 4(a)** | $x - 3y + z = 0$ |

*(Show your working below and on the next page.)*

$$(2,1,1) \times (1,1,2) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= \vec{i} - 3\vec{j} + \vec{k}$$

$$\therefore \quad \underline{\underline{x - 3y + z = 0}}$$

**Question 4 (b)** [5 marks]

A space curve $C$ is defined by the vector parametric equation

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k}.$$

Let $L$ denote the tangent line to $C$ at the point corresponding to $t = 1$. Find the distance from the origin $(0, 0, 0)$ to this line $L$. Give **exact value** for your answer.

| | |
|---|---|
| **Answer 4(b)** | $\sqrt{\frac{3}{7}}$ |

*(Show your working below and on the next page.)*

$$\vec{r}\,'(t) = \vec{i} + 2t\vec{j} - 3t^2\vec{k}$$

$$\vec{r}(1) = \vec{i} + \vec{j} - \vec{k}$$

$$\vec{r}\,'(1) = \vec{i} + 2\vec{j} - 3\vec{k}$$

$$\vec{r}(1) \times \vec{r}\,'(1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & 2 & -3 \end{vmatrix} = -\vec{i} + 2\vec{j} + \vec{k}$$

$$\text{distance} = \frac{\|\vec{r}(1) \times \vec{r}\,'(1)\|}{\|\vec{r}\,'(1)\|} = \sqrt{\frac{1+4+1}{1+4+9}} = \underline{\underline{\sqrt{\frac{3}{7}}}}$$

**Question 5 (a)** [5 marks]

Let $f(x, y, z)$ be a differentiable function of three variables and

$$f(1,2,3) = 2011, \quad \frac{\partial f}{\partial x}(1,2,3) = -2, \quad \frac{\partial f}{\partial y}(1,2,3) = 3.$$

It was found that if the point Q moved from $(1,2,3)$ a distance 0.2 unit towards $(3,6,7)$, the value of $f$ became 2012. Estimate the value of $\frac{\partial f}{\partial z}(1,2,3)$.

| Answer 5(a) | $5.5$ (or $\frac{11}{2}$) |
| --- | --- |

*(Show your working below and on the next page.)*

Let $\frac{\partial f}{\partial z}(1,2,3) = a$

$$\vec{u} = \frac{(3,6,7)-(1,2,3)}{\|(3,6,7)-(1,2,3)\|} = \frac{(2,4,4)}{6} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$0.2\left(\frac{1}{3}(-2) + \frac{2}{3}(3) + \frac{2}{3}a\right) \simeq 2012 - 2011 = 1$$

$$-2 + 6 + 2a \simeq 15$$

$$a \simeq \frac{11}{2} = \underline{\underline{5.5}}$$

**Question 5 (b)** [5 marks]

Let $h(x)$ and $g(x)$ be two twice differentiable functions of one variable $x$. Let $f(x, y)$ be a function of two variables $x$ and $y$ defined by

$$f(x, y) = yh(x) + g(x).$$

It is known that

$$h(2) = 3, \quad h(3) = 0, \quad h'(2) = -5, \quad h'(3) = -1,$$

$$g(2) = 1, \quad g(3) = 3, \quad g'(2) = 0, \quad \text{and} \quad g'(3) = 6.$$

Show that $f$ has at least one critical point. Find this critical point and determine whether it is a local maximum, local minimum or a saddle point.

| **Answer 5(b)** | $(3,6)$ ~ SADDLE POINT |
|---|---|

*(Show your working below and on the next page.)*

$f_x = y h'(x) + g'(x)$

$f_y = h(x)$

$x=3,\ y=6 \Rightarrow f_x = 6h'(3) + g'(3) = 6(-1) + 6 = 0$

$f_y = h(3) = 0$

$\therefore$ $(3,6)$ is a critical point

$f_{xy} = h'(x), \quad f_{yy} = 0$

$\therefore f_{xx} f_{yy} - f_{xy}^2 = -[h'(x)]^2$

$= -(-1)^2 = -\text{ve}$ when $x=3$

$\therefore (3,6)$ is a SADDLE POINT

**Question 6 (a)** [5 marks]

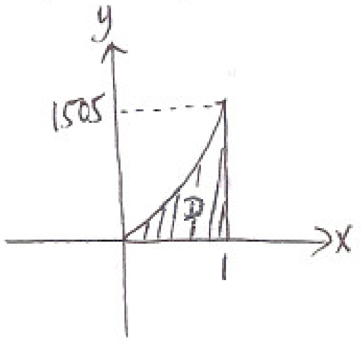
Find the **exact value** of the double integral

$$\int\int_D e^{x^{1505}}\,dx dy,$$

where $D$ is the finite domain bounded by the curve $y = 1505x^{1504}$ and the two lines: $y = 0$, $x = 1$.

| Answer 6(a) | $e - 1$ |
| --- | --- |

*(Show your working below and on the next page.)*




$$\iint_D e^{x^{1505}}\,dx\,dy$$

$$= \int_0^1 \int_0^{1505x^{1504}} e^{x^{1505}}\,dy\,dx$$

$$= \int_0^1 1505x^{1504} e^{x^{1505}}\,dx$$

$$= e^{x^{1505}}\Big|_0^1$$

$$= \underline{\underline{e-1}}$$

## Question 6 (b) [5 marks]

The region $R$ lies above the paraboloid

$$z = 5 - x^2 - y^2$$

and below the paraboloid

$$z = 9 - 2x^2 - 2y^2$$

Find the **exact value** of the volume of $R$, giving your answer in terms of $\pi$.

| Answer 6(b) | $8\pi$ |
|---|---|

*(Show your working below and on the next page.)*

The two paraboloids intersect at

$$5 - x^2 - y^2 = 9 - 2x^2 - 2y^2$$
$$\Rightarrow x^2 + y^2 = 4$$

$$\text{Vol of } R = \iint_{x^2+y^2 \le 4} (9 - 2x^2 - 2y^2) - (5 - x^2 - y^2)\, dx\,dy$$

$$= \iint_{x^2+y^2 \le 4} (4 - x^2 - y^2)\, dx\,dy$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2)\, r\,dr\,d\theta$$

$$= 2\pi \left[ 2r^2 - \frac{1}{4} r^4 \right]_0^2$$

$$= \underline{\underline{8\pi}}$$

**Question 7 (a)** [5 marks]

Find the **exact value** of the line integral

$$\int_C \sqrt{z}ds,$$

where $C$ is the space curve given by

$$x = t\cos t^2, \quad y = t\sin t^2, \quad z = t^2,$$

with $0 \le t \le 2.$

| Answer 7(a) | 10 |
|---|---|

*(Show your working below and on the next page.)*

$$\vec{r}(t) = (t\cos t^2,\ t\sin t^2,\ t^2)$$

$$\vec{r}\,'(t) = (\cos t^2 - 2t^2\sin t^2,\ \sin t^2 + 2t^2\cos t^2,\ 2t)$$

$$\|\vec{r}\,'(t)\| = \sqrt{1 + 4t^4 + 4t^2} = 2t^2+1$$

$$\int_C \sqrt{z}\,ds = \int_0^2 t(2t^2+1)\,dt$$

$$= \int_0^2 (2t^3+t)\,dt$$

$$= \left[\tfrac{1}{2}t^4 + \tfrac{1}{2}t^2\right]_0^2 = \underline{\underline{10}}$$

**Question 7 (b)** [5 marks]

Using Green's Theorem, or otherwise, find the **exact value** of the line integral

$$\oint_C (y + e^{\sqrt{1+x^5}})dx + (3x + e^{\cos y^2})dy,$$

where $C$ is the boundary of the square with vertices at $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$ oriented in the counterclockwise sense when viewed from above.

| **Answer 7(b)** | $8$ |
|---|---|

*(Show your working below and on the next page.)*

$$\oint_C (y+e^{\sqrt{1+x^5}})\,dx + (3x+e^{\cos y^2})\,dy$$

$$= \iint_{\text{square}} (3-1)\,dx\,dy$$

$$= 2\,(\text{area of square})$$

$$= \underline{\underline{8}}$$

## Question 8 (a) [5 marks]

Find the **exact value** of the surface integral

$$\iint_S y^2 z \, dS,$$

where $S$ is the portion of the cylinder $x^2 + y^2 = 4$ lying between the two planes $z = 3$ and $z = 0$.

| | |
|---|---|
| **Answer** 8(a) | $36\pi$ |

*(Show your working below and on the next page.)*

$S: \vec{r}(u,v) = 2\cos u\,\vec{i} + 2\sin u\,\vec{j} + v\,\vec{k}, \quad 0 \le u \le 2\pi, \ 0 \le v \le 3.$

$\vec{r}_u = -2\sin u\,\vec{i} + 2\cos u\,\vec{j} + 0\,\vec{k}$

$\vec{r}_v = 0\,\vec{i} + 0\,\vec{j} + \vec{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\cos u\,\vec{i} + 2\sin u\,\vec{j}$$

$\|\vec{r}_u \times \vec{r}_v\| = 2$

$$\iint_S y^2 z\,dS = \int_0^3 \int_0^{2\pi} (4\sin^2 u)\, v\,(2\,du\,dv)$$

$$= \left[ v^2 \right]_{v=0}^{3} \, 2\int_0^{2\pi} (1 - \cos 2u)\,du$$

$$= \underline{\underline{36\pi}}$$

**Question 8 (b)** [5 marks]

Use Stokes' Theorem to find the **exact value** of the line integral

$$\oint_C \left( y\, dx + z^2\, dy + x\, dz \right),$$

where $C$ is the curve of intersection of the plane $2x + z = 0$ and the ellipsoid $x^2 + 5y^2 + z^2 = 1$, oriented counterclockwise as seen from above.

| **Answer** **8(b)** | $-\frac{\pi}{5}$ |
|---|---|

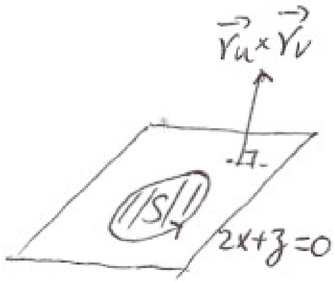
*(Show your working below and on the next page.)*

$$C: \begin{cases} 2x+z=0 \\ x^2+5y^2+z^2=1 \end{cases} \Rightarrow x^2+5y^2+4x^2=1 \Rightarrow 5x^2+5y^2=1$$

Let $S$ = region on $2x+z=0$ bounded by $C$.

$\therefore S: \vec{r}(u,v) = u\vec{i} + v\vec{j} - 2u\vec{k}, \quad u^2+v^2 \leq \left(\frac{1}{\sqrt{5}}\right)^2$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 2\vec{i} + \vec{k}$$




$\vec{r}_u \times \vec{r}_v$ points up $\Rightarrow$ correct orientation

$\vec{F} = y\vec{i} + z^2\vec{j} + x\vec{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z^2 & x \end{vmatrix} = -2z\vec{i} - \vec{j} - \vec{k}$$

*(More working space for Question 8(b))*

Stoke's theorem

$$\Rightarrow \oint_C y\,dx + z^2\,dy + x\,dz = \iint_{u^2+v^2 \le \left(\frac{1}{\sqrt{5}}\right)^2} (8u-1)\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{5}}} (8r\cos\theta - 1)\,r\,dr\,d\theta$$

$$= \int_0^{\frac{1}{\sqrt{5}}}\int_0^{2\pi} (8r^2\cos\theta - r)\,d\theta\,dr$$

$$= 2\pi\left[-\frac{1}{2}r^2\right]_0^{\frac{1}{\sqrt{5}}}$$

$$= -\frac{\pi}{5}$$

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**END OF PAPER**