Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2010-2011

MA1505 MATHEMATICS I

November 2010 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
(a)								
(b)								

Question 1 (a) [5 marks]

Find the slope of the tangent to the curve

$$y = x + \frac{1}{x}$$

when $x = 2$.

Answer 1(a)	$\frac{3}{4}$

(Show your working below and on the next page.)

$$y' = 1 - \frac{1}{x^2}$$

$$x = 2 \Rightarrow y' = 1 - \frac{1}{4} = \underline{\underline{\frac{3}{4}}}$$

Question 1 (b) [5 marks]

Find the **exact value** of each of the following limits:

(i) $$\lim_{x\to(-1)} \frac{1+x^{23}}{1-x^2}.$$

(ii) $$\lim_{x\to 0} \frac{\sin^3\left(e^{101x}-1\right)}{e^{101x}(\sin 100x)(\sin 101x)(\sin 102x)}.$$

Answer 1(b)(i)	$\frac{23}{2}$	**Answer 1(b)(ii)**	$\frac{101^2}{10200} = \frac{10201}{10200}$

(Show your working below and on the next page.)

(i) limit $= \lim_{x\to -1} \frac{23x^{22}}{-2x} = \underline{\underline{\frac{23}{2}}}$

(ii) limit

$$= \left(\lim_{x\to 0} \frac{1}{e^{101x}}\right)\left(\lim_{x\to 0} \frac{\sin(e^{101x}-1)}{e^{101x}-1}\right)^3 \left(\lim_{x\to 0} \frac{e^{101x}-1}{\sin 100x}\right)\left(\lim_{x\to 0} \frac{e^{101x}-1}{\sin 101x}\right)\left(\lim_{x\to 0} \frac{e^{101x}-1}{\sin 102x}\right)$$

$$= \left(\lim_{x\to 0} \frac{101e^{101x}}{100\cos 100x}\right)\left(\lim_{x\to 0} \frac{101e^{101x}}{101\cos 101x}\right)\left(\lim_{x\to 0} \frac{101e^{101x}}{102\cos 102x}\right)$$

$$= \frac{101^2}{10200} = \underline{\underline{\frac{10201}{10200}}}$$

Question 2 (a) [5 marks]

Let $f(t)$ be a differentiable function such that its derivative $f'(t)$ is continuous. We do not know the formula for f, but we know the following values:

$$f(0) = 1, f(1) = 2, f(2) = 0, f(3) = 3,$$

$$f'(0) = -2, f'(1) = -1, f'(2) = -3, f'(3) = 2.$$

Let $F(x)$ be the function defined by

$$F(x) = \left(x^2 \int_0^x f'(t)dt\right) + \left(\int_0^x t^2 f'(t)dt\right).$$

Find the **exact value** of $F'(3)$.

Answer 2(a)	48

(Show your working below and on the next page.)

$$F(x) = x^2\int_0^x f'(t)dt + \int_0^x t^2 f'(t)dt$$

$$= x^2\{f(x) - f(0)\} + \int_0^x t^2 f'(t)dt$$

$$F'(x) = 2x\{f(x) - f(0)\} + x^2 f'(x) + x^2 f'(x)$$

$$= 2x\{f(x) - f(0)\} + 2x^2 f'(x)$$

$$F'(3) = 6\{f(3) - f(0)\} + 18 f'(3)$$

$$= 6\{3 - 1\} + (18)(2)$$

$$= \underline{\underline{48}}$$

Question 2 (b) [5 marks]

Let

$$f(x) = \int_0^{x^2} e^{-t^2}\,dt.$$

Find the **exact value** of

$$f^{(2010)}(0).$$

Express your answer in terms of factorials.

Answer 2(b)	$\dfrac{2010!}{(1005)(502!)}$

(Show your working below and on the next page.)

$$f(x) = \sum_{n=0}^{\infty} \int_0^{x^2} \frac{(-1)^n t^{2n}}{n!}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(n!)} x^{4n+2}$$

$$2010 = 4(502) + 2$$

$$\frac{f^{(2010)}(0)}{2010!} = \frac{(-1)^{502}}{(1005)(502!)}$$

$$f^{(2010)}(0) = \frac{2010!}{(1005)(502!)}$$

Question 3 (a) [5 marks]

Let $f(x)$ be a function defined by

$$f(x) = \cos\frac{x}{2} \quad \text{if } -\pi < x < \pi\ ,$$

and $f(x + 2\pi) = f(x)$.
Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series for $f(x)$.
(i) Find the **exact value** of a_0.
(ii) Find the **exact value** of $\pi(a_9)$. Give your answer as a fraction in its simplest form.

Answer 3(a)(i)	$\frac{2}{\pi}$	Answer 3(a)(ii)	$\frac{4}{323}$

(Show your working below and on the next page.)

(i) $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\frac{x}{2}\,dx = \frac{1}{\pi}\int_0^{\pi}\cos\frac{x}{2}\,dx = \frac{2}{\pi}\sin\frac{x}{2}\Big|_0^{\pi} = \underline{\underline{\frac{2}{\pi}}}$

(ii) $a_9 = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\frac{x}{2}\cos 9x\,dx =$

$= \frac{1}{\pi}\int_0^{\pi}\left(\cos\frac{19}{2}x + \cos\frac{17}{2}x\right)dx$

$= \frac{1}{\pi}\left[\frac{2}{19}\sin\frac{19}{2}x + \frac{2}{17}\sin\frac{17}{2}x\right]_0^{\pi} = \frac{1}{\pi}\left(-\frac{2}{19} + \frac{2}{17}\right) = \frac{4}{323\pi}$

$\therefore \pi a_9 = \underline{\underline{\frac{4}{323}}}$

Question 3 (b) [5 marks]

Let

$$f(x) = x + 1, \quad 0 \le x \le 2.$$

Write down the sine Fourier half range expansion for $f(x)$ up to and including the first two non-zero terms. Give **exact values** in terms of π in the simplest form for your answer.

Answer 3(b)	$\frac{8}{\pi}\sin\frac{\pi x}{2} - \frac{2}{\pi}\sin \pi x$

(Show your working below and on the next page.)

$$b_n = \frac{2}{2}\int_0^2 (x+1)\sin\frac{n\pi x}{2}\,dx$$

$$= -\frac{2}{n\pi}\int_0^2 (x+1)\,d\left(\cos\frac{n\pi x}{2}\right)$$

$$= \left[-\frac{2}{n\pi}(x+1)\cos\frac{n\pi x}{2}\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi x}{2}\,dx$$

$$= -\frac{6}{n\pi}\cos n\pi + \frac{2}{n\pi} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_0^2$$

$$= -\frac{6}{n\pi}(-1)^n + \frac{2}{n\pi}$$

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} = \underline{\underline{\frac{8}{\pi}\sin\frac{\pi x}{2} - \frac{2}{\pi}\sin\pi x + \cdots}}$$

Question 4 (a) [5 marks]

Let

$$
\begin{aligned}
\mathbf{u} &= \mathbf{i} - 6\mathbf{j} - \mathbf{k} \\
\mathbf{v} &= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \\
\mathbf{w} &= 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}
\end{aligned}
$$

be three vectors. Find the **exact expression** of the projection of $\mathbf{u}$ onto the vector $\mathbf{v} \times \mathbf{w}$.

Answer 4(a)	$2i - 4j + 2k$

(Show your working below and on the next page.)

$$V \times W = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = -4i + 8j - 4k$$

$$\text{Proj}_{V \times W} u = \frac{u \cdot (V \times W)}{\| V \times W \|^2} (V \times W)$$

$$= \frac{-4 - 48 + 4}{16 + 64 + 16} (-4i + 8j - 4k)$$

$$= \underline{\underline{2i - 4j + 2k}}$$

Question 4 (b) [5 marks]

A space curve C is defined by the vector parametric equation

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} - t\mathbf{k}.$$

Let L denote the tangent line to C at the point corresponding to $t = 2$. Let S be the plane $2x - y + z = 7$. Find the coordinates of the point of intersection of L and S. Give **exact values** for your answer.

Answer 4(b)	$(-1, -8, 1)$

(Show your working below and on the next page.)

$$r'(t) = i + 2tj - k$$

$$r'(2) = i + 4j - k$$

$$r(2) = 2i + 4j - 2k$$

$$L: \tilde{r}(t) = (2+t,\ 4+4t,\ -2-t)$$

$$\therefore\ 2(2+t) - (4+4t) + (-2-t) = 7$$

$$-2 - 3t = 7$$

$$t = -3$$

$$\underline{\underline{(-1, -8, 1)}}$$

Question 5 (a) [5 marks]

Let

$$f(x, y, z) = x^2yz.$$

Find the **exact value** of the directional derivative of f at the point $(2, -1, 3)$ in the direction of the vector $\mathbf{u} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Answer 5(a)	$\frac{28}{3}$

(Show your working below and on the next page.)

$$\nabla f = 2xyz\, i + x^2 z\, j + x^2 y\, k$$

$$\nabla f(2,-1,3) = -12 i + 12 j - 4k$$

$$v = \frac{u}{\|u\|} = \frac{1}{3}(-i + 2j + 2k)$$

$$D_v f(2,-1,3) = \nabla f(2,-1,3) \cdot v$$

$$= \frac{1}{3}(12 + 24 - 8)$$

$$= \underline{\underline{\frac{28}{3}}}$$

Question 5 (b) [5 marks]

Find and classify all the critical points of the function

$$f(x, y) = y^3 + x^2 - 6xy + 2010.$$

Answer 5(b)	$(0,0)$ saddle $(18,6)$ loc. min.

(Show your working below and on the next page.)

$f_x = 2x - 6y = 0 \Rightarrow x = 3y$

$f_y = 3y^2 - 6x = 0 \Rightarrow y^2 = 2x$

$\therefore y^2 = 6y \Rightarrow y = 0, 6$

$\therefore$ Two critical points : $(0,0)$, $(18,6)$

$f_{xx} = 2$, $f_{xy} = f_{yx} = -6$, $f_{yy} = 6y$

	f_{xx}	$f_{xx}f_{yy} - f_{xy}^2$	
$(0,0)$	$+$	$-$	saddle
$(18,6)$	$+$	$+$	loc. min.

Question 6 (a) [5 marks]

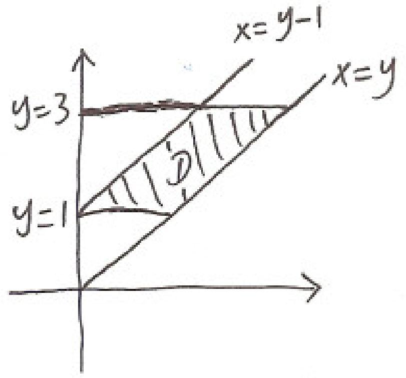
Find the **exact value** of the double integral

$$\int\int_D (x^2 + y^2)\, dx dy,$$

where D is the parallelogram bounded by the four lines: $y = x$, $y = x + 1$, $y = 1$ and $y = 3$.

Answer 6(a)	14

(Show your working below and on the next page.)



$$\iint_D (x^2+y^2)\,dxdy$$

$$= \int_1^3 \int_{y-1}^{y} (x^2+y^2)\,dx\,dy$$

$$= \int_1^3 \left[\frac{1}{3}x^3 + xy^2\right]_{x=y-1}^{x=y} dy$$

$$= \int_1^3 \left[\frac{1}{3}y^3 + y^3 - \frac{1}{3}(y-1)^3 - (y-1)y^2\right] dy$$

$$= \left[\frac{1}{3}y^4 - \frac{1}{12}(y-1)^4 - \frac{1}{4}y^4 + \frac{1}{3}y^3\right]_1^3$$

$$= \frac{81}{12} - \frac{16}{12} + 9 - \frac{1}{12} - \frac{1}{3}$$

$$= \underline{\underline{14}}$$

Question 6 (b) [5 marks]

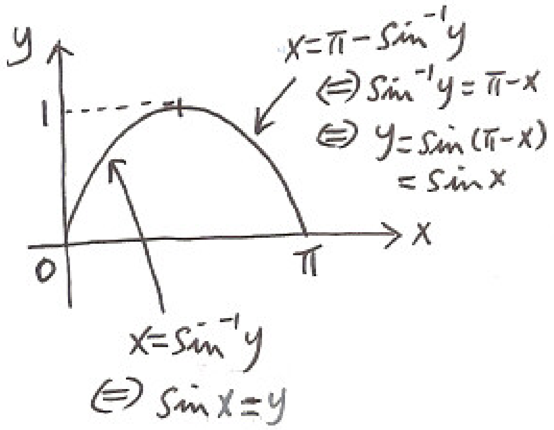
Find the **exact value** of the iterated integral

$$\int_0^1 \left[\int_{\sin^{-1} y}^{\pi - \sin^{-1} y} x^2 dx \right] dy \,,$$

where $-\frac{\pi}{2} \le \sin^{-1} y \le \frac{\pi}{2}$.

Answer 6(b)	$\pi^2 - 4$

(Show your working below and on the next page.)



$$\int_0^1 \int_{\sin^{-1} y}^{\pi - \sin^{-1} y} x^2 dx\, dy$$

$$= \int_0^{\pi} \int_0^{\sin x} x^2 dy\, dx$$

$$= \int_0^{\pi} \left[x^2 y \right]_{y=0}^{y=\sin x} dx$$

$$= \int_0^{\pi} x^2 \sin x\, dx = -\int_0^{\pi} x^2 d(\cos x)$$

$$= -x^2 \cos x \Big|_0^{\pi} + 2\int_0^{\pi} x \cos x\, dx$$

$$= \pi^2 + 2x \sin x \Big|_0^{\pi} - 2\int_0^{\pi} \sin x\, dx$$

$$= \pi^2 + 2\cos x \Big|_0^{\pi} = \underline{\underline{\pi^2 - 4}}$$

Question 7 (a) [5 marks]

Find the **exact value** of the line integral

$$\int_C (xyz)\,ds\ ,$$

where C is the part of the circular helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 3t\mathbf{k}$ from $(1, 0, 0)$ to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)$.

Answer 7(a)	$\frac{3\sqrt{10}}{8}$

(Show your working below and on the next page.)

$$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + 3\,\vec{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 9} = \sqrt{10}$$

$$\int_C (xyz)\,ds = \int_0^{\frac{\pi}{4}} 3t\cos t\sin t\,\sqrt{10}\,dt$$

$$= \frac{3\sqrt{10}}{2}\int_0^{\pi/4} t\sin 2t\,dt$$

$$= -\frac{3\sqrt{10}}{4}\int_0^{\pi/4} t\,d(\cos 2t)$$

$$= -\frac{3\sqrt{10}}{4}\,t\cos 2t\Big|_0^{\pi/4} + \frac{3\sqrt{10}}{4}\int_0^{\pi/4}\cos 2t\,dt$$

$$= \frac{3\sqrt{10}}{8}\sin 2t\Big|_0^{\pi/4} = \underline{\underline{\frac{3\sqrt{10}}{8}}}$$

Question 7 (b) [5 marks]

Find the **exact value** of the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F} = (y - x)\,\mathbf{i} + (z - x)\,\mathbf{j} + (x - y)\,\mathbf{k}$, and C is the curve of intersection of the plane

$$x + z = 2$$

and the cylinder

$$x^2 + y^2 = 4,$$

oriented in the counterclockwise sense when viewed from above.

Answer 7(b)	-16π

(Show your working below and on the next page.)

$x = 2\cos t$, $y = 2\sin t \Rightarrow z = 2 - x = 2 - 2\cos t$

$\therefore C: \vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + (2 - 2\cos t)\vec{k}, \quad 0 \le t \le 2\pi.$

$\frac{d\vec{r}}{dt} = -2\sin t\,\vec{i} + 2\cos t\,\vec{j} + 2\sin t\,\vec{k}$

$\vec{F} = (2\sin t - 2\cos t)\vec{i} + (2 - 4\cos t)\vec{j} + (2\cos t - 2\sin t)\vec{k}$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left[-2\sin t(2\sin t - 2\cos t) + 2\cos t(2 - 4\cos t) + 2\sin t(2\cos t - 2\sin t)\right] dt$$

$$= \int_0^{2\pi} (4\cos t + 4\sin 2t - 8)\, dt$$

$$= \left[4\sin t - 2\cos 2t - 8t\right]_0^{2\pi} = \underline{\underline{-16\pi}}$$

Question 8 (a) [5 marks]

Find the **exact value** of the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

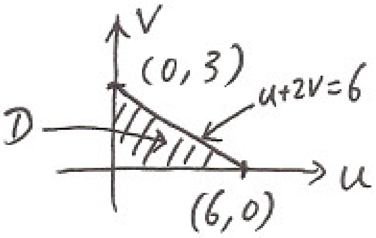
where $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S is the portion of the plane $x + 2y + 3z = 6$ in the first octant. The orientation of S is given by the downward normal vector.

Answer 8(a)	-33

(Show your working below and on the next page.)

$x = u,\ y = v \Rightarrow u + 2v + 3z = 6 \Rightarrow z = \frac{1}{3}(6 - u - 2v)$

$\vec{r}(u,v) = u\vec{i} + v\vec{j} + \frac{1}{3}(6 - u - 2v)\vec{k}$



$\vec{r}_u = \vec{i} + 0\vec{j} - \frac{1}{3}\vec{k}$

$\vec{r}_v = 0\vec{i} + \vec{j} - \frac{2}{3}\vec{k}$

$\vec{r}_u \times \vec{r}_v = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \vec{k}$ = upward normal

$\therefore$ orientation $= -\vec{r}_u \times \vec{r}_v = -\frac{1}{3}\vec{i} - \frac{2}{3}\vec{j} - \vec{k}$

$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (-\vec{r}_u \times \vec{r}_v)\, du\, dv = \iint_D \left[-\frac{1}{3}u^2 - \frac{2}{3}v^2 - \frac{1}{9}(6-u-2v)^2\right] du\, dv$

$= \int_0^3 \int_0^{6-2v} \left(-\frac{4}{9}u^2 - \frac{10}{9}v^2 - 4 + \frac{4}{3}u + \frac{8}{3}v - \frac{4}{9}uv\right) du\, dv$

$= \underline{\underline{-33}}$

Question 8 (b) [5 marks]

Use Stokes' Theorem to find the **exact value** of the surface integral

$$\int\int_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S},$$

where $\mathbf{F} = -yz\mathbf{i} + x\mathbf{j} - e^x(\sin y)[\cos(z^2)]\mathbf{k}$, and S is the part of the elliptical paraboloid

$$z = x^2 + 4y^2$$

for which $z \leq 1$. The orientation of S is given by the outward normal vector.

Answer 8(b)	$-\pi$

(Show your working below and on the next page.)

Boundary of $S = C$: $\vec{r}(t) = \cos t\,\vec{i} + \frac{1}{2}\sin t\,\vec{j} + \vec{k}$, $0 \leq t \leq 2\pi$.

$\therefore \vec{r}'(t) = -\sin t\,\vec{i} + \frac{1}{2}\cos t\,\vec{j}$

$\because$ orientation of S = outward normal

$\therefore$ orientation of S is compatible to orientation of $(-C)$.

$$\therefore \iint_S (\text{Curl } \vec{F}) \cdot d\vec{S} = \int_{-C} \vec{F} \cdot d\vec{r}$$

$$= -\int_0^{2\pi} \left(\frac{1}{2}\sin^2 t + \frac{1}{2}\cos^2 t\right) dt$$

$$= \underline{\underline{-\pi}}$$