2010/2011 SEMESTER 1 MID-TERM TEST

MA1505 MATHEMATICS I

29 September 2010

8:30pm to 9:30pm

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY:

1. This test paper consists of **TEN (10)** multiple choice questions and comprises **Twelve (13)** printed pages.

2. Answer all 10 questions. 1 mark for each correct answer. No penalty for wrong answers. Full mark is 10.

3. All answers (Choices A, B, C, D, E) are to be submitted using the pink form (FORM CC1/10).

4. Use **only 2B pencils** for FORM CC1/10.

5. On FORM CC1/10 (section B for matric numbers starting with A, section C for others), **write** your **matriculation number** and **shade** the corresponding numbered circles **completely**. Your FORM CC1/10 will be graded by a computer and it will record a **ZERO** for your score if your matriculation number is not correct.

6. **Write your full name** in the blank space for module code in section A of FORM CC1/10.

7. Only circles for answers 1 to 10 are to be shaded.

8. For each answer, the circle corresponding to your choice should be **properly** and **completely** shaded. If you change your answer later, you must make sure that the original answer is properly erased.

9. For each answer, **do not shade more than one circle**. The answer for a question with more than one circle shaded will be marked wrong.

10. **Do not fold** FORM CC1/10.

11. Submit FORM CC1/10 before you leave the test hall.

Formulae List

1. The **Taylor series** of f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

2.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

4.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

5.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

6.

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

1. Let $y = \frac{1}{1+x^2}$, and $x = \cot\theta$. Find $\frac{dy}{dx}$ and express your answer in terms of θ.

(A) $-\sin^2\theta\sin 2\theta$

(B) $2\sin\theta\cos\theta$

(C) $\sin\theta\cos 2\theta$

(D) $-\sin\theta\sin 2\theta$

(E) None of the above

2. A light shines from the top of a lamp post 20 m high. A particle is projected upwards from the ground at a point 10 m away from the lamp post. It is known that the particle covers a distance

$$s = 20t - 5t^2$$

in t seconds, where s is measured in metre. Find the speed of the shadow of the particle on the ground 1 second later.

(A) 20 m per second

(B) 40 m per second

(C) 60 m per second

(D) 80 m per second

(E) None of the above

3. A curve is defined implicitly by the equation

$$x^2 + xy + y^2 - x = 2.$$

Let L denote the tangent line to this curve at the point $(2, -2)$.

Find the x-coordinate of the point of intersection of L with the line $y = -\frac{1}{2}$.

(A) 8

(B) 3

(C) $\frac{5}{2}$

(D) $-\frac{5}{2}$

(E) None of the above

4. Let a be a positive constant. Let M and m denote the absolute maximum value and absolute minimum value respectively of the function

$$f(x) = x^3 - 3a^2x - a^3,$$

in the domain $\left[-\frac{3a}{2}, \frac{a}{2}\right]$. Find $\frac{M}{m}$.

(A) $\frac{1}{8}$

(B) $-\frac{1}{19}$

(C) $-\frac{19}{24}$

(D) $-\frac{8}{19}$

(E) None of the above

5. Suppose $0 < x < \frac{\pi}{2}$. Then

$$\int \left(\sec^2 x\right) \ln\left(\tan x\right)\ dx =$$

(A) $(\tan x)\ln(\tan x) + C$

(B) $\ln\left(\frac{\tan x}{e}\right)^{\tan x} + C$

(C) $\left(\frac{\sec^2 x}{e}\right)\ln(\tan x) + C$

(D) $(\tan x)\ln(\tan x) - x + C$

(E) None of the above

6. Let a be a positive constant. Find the area of the finite region bounded by the curves $y^2 = x + 4a^2$ and $x - ay + 2a^2 = 0$.

(A) $\frac{13}{6}a^3$

(B) $6a^3$

(C) $\frac{9}{2}a^3$

(D) $\frac{11}{3}a^3$

(E) None of the above

7. Find the exact value of

$$\int_0^{\pi} |x \cos x| \, dx.$$

(A) π

(B) 2

(C) $\frac{\pi}{2}$

(D) $\frac{1}{2}$

(E) None of the above

8. A finite region R is bounded by the curve $y = \sqrt{\tan x}$, and the lines $x = \frac{\pi}{4}$ and $y = 0$. Find the volume of the solid formed by revolving R one complete round about the x-axis.

(A) $\pi \ln 2$

(B) $\frac{\pi^2}{4}$

(C) $\frac{1}{2} \ln (2\pi)$

(D) $\left(\ln \sqrt{2}\right)^{\pi}$

(E) None of the above

9. Evaluate the sum

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1+(-1)^{n+1}+(-2)^{n}}{2^{n+1}}\right).$$

(A) $\sqrt{e}+\frac{1}{\sqrt{e}}+\frac{1}{e}$

(B) $\sqrt{e}-\frac{1}{\sqrt{e}}+\frac{1}{e}$

(C) $\frac{\sqrt{e}}{2}+\frac{1}{2\sqrt{e}}+\frac{1}{2e}$

(D) $\frac{\sqrt{e}}{2}-\frac{1}{2\sqrt{e}}+\frac{1}{e}$

(E) None of the above

10. Let

$$\sum_{n=0}^{\infty} a_n (x+3)^n$$

denote the Taylor Series of $\frac{1}{2-x}$ at the point $x = -3$.

Then $a_5 =$

(A) 1

(B) $\frac{1}{3125}$

(C) $\frac{1}{15\,625}$

(D) $\frac{1}{625}$

(E) None of the above

END OF PAPER

Additional blank page for you to do your calculations

National University of Singapore
Department of Mathematics

2010-2011 Semester 1   MA1505 Mathematics I   Mid-Term Test Answers

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Question	1	2	3	4	5	6	7	8	9	10
Answer	A	D	E	D	B	C	A	E	E	C

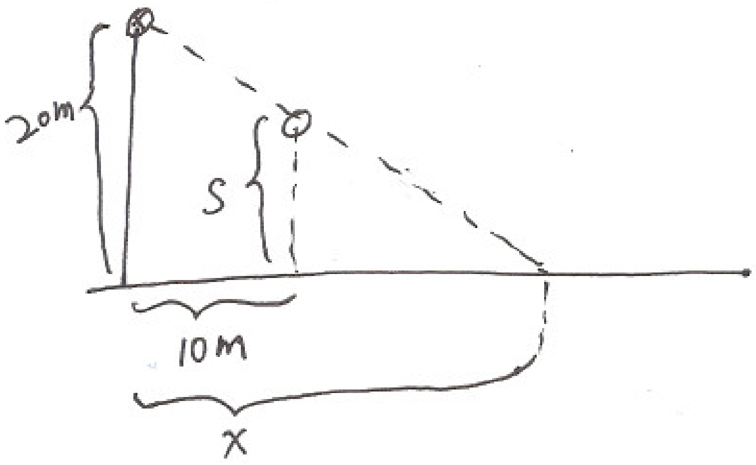
2010 mid-Term Test solutions

1). A

$$y = \frac{1}{1+x^2} = \frac{1}{1+\cot^2\theta} = \frac{1}{\csc^2\theta} = \sin^2\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\sin\theta\cos\theta}{-\csc^2\theta} = -\sin^2\theta\sin 2\theta$$

2). D



$$\frac{x-10}{x} = \frac{S}{20} = \frac{20t-5t^2}{20} = t - \frac{1}{4}t^2$$

$$x - 10 = \left(t - \frac{1}{4}t^2\right)x$$

$$x = \frac{10}{\frac{1}{4}t^2 - t + 1} = \frac{40}{t^2-4t+4} = \frac{40}{(t-2)^2}$$

$$\frac{dx}{dt} = -\frac{80}{(t-2)^3}$$

$$t=1 \Rightarrow \frac{dx}{dt} = -\frac{80}{(-1)^3} = 80$$

3). E

$$x^2+xy+y^2-x=2$$

$$2x+xy'+y+2yy'-1=0$$

$$x=2,\ y=-2 \Rightarrow 4+2y'-2-4y'-1=0 \Rightarrow y'=\frac{1}{2}$$

$$L:\ y+2=\frac{1}{2}(x-2)$$

$$y=-\frac{1}{2} \Rightarrow 3=x-2 \Rightarrow x=\underline{\underline{5}}$$

4). D

$$f(x)=x^3-3a^2x-a^3$$

$$f'(x)=3x^2-3a^2=3(x+a)(x-a)$$

$$f\left(-\frac{3a}{2}\right)=-\frac{27}{8}a^3+\frac{9}{2}a^3-a^3=\frac{1}{8}a^3$$

$$f(-a)=-a^3+3a^3-a^3=a^3$$

$$f\left(\frac{a}{2}\right)=\frac{1}{8}a^3-\frac{3}{2}a^3-a^3=-\frac{19}{8}a^3$$

$$\therefore M=a^3,\quad m=-\frac{19}{8}a^3$$

$$\frac{M}{m}=\underline{\underline{-\frac{8}{19}}}$$

5). B

$$\int \sec^2 x \ln(\tan x)\,dx=\int \ln(\tan x)\,d(\tan x)$$

$$=\tan x \ln(\tan x)-\int \tan x \frac{1}{\tan x}\,d(\tan x)$$

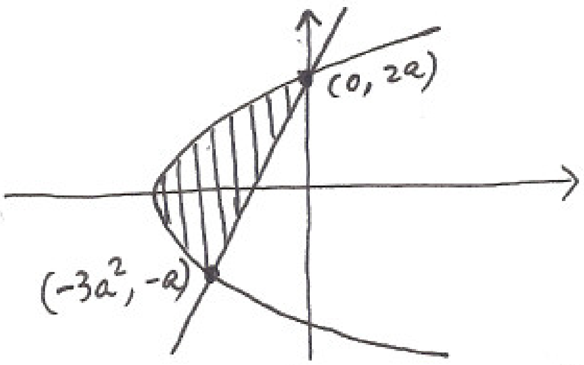
$$=\tan x\{\ln(\tan x)-1\}+C=(\tan x)\ln\left(\frac{\tan x}{e}\right)+C$$

$$=\underline{\underline{\ln\left(\frac{\tan x}{e}\right)^{\tan x}+C}}$$

6). C

$$\left.\begin{aligned} y^2 &= x + 4a^2 \\ x - ay + 2a^2 &= 0 \end{aligned}\right\} \Rightarrow y^2 - 4a^2 - ay + 2a^2 = 0$$

$$\Rightarrow y^2 - ay - 2a^2 = 0 \Rightarrow (y-2a)(y+a) = 0$$



$$\text{Area} = \int_{-a}^{2a} \left[(ay - 2a^2) - (y^2 - 4a^2)\right] dy$$

$$= \left[\tfrac{1}{2}ay^2 + 2a^2y - \tfrac{1}{3}y^3\right]_{-a}^{2a}$$

$$= a^3\left\{\left(2 + 4 - \tfrac{8}{3}\right) - \left(\tfrac{1}{2} - 2 + \tfrac{1}{3}\right)\right\} = \underline{\underline{\tfrac{9}{2}a^3}}$$

7). A

Let $F(x) = \int x\cos x\,dx$

$= \int x\,d(\sin x)$

$= x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x$

$$\int_0^{\pi} |x\cos x|\,dx = \int_0^{\frac{\pi}{2}} |x\cos x|\,dx + \int_{\frac{\pi}{2}}^{\pi} |x\cos x|\,dx$$

$$= \int_0^{\frac{\pi}{2}} x\cos x\,dx - \int_{\frac{\pi}{2}}^{\pi} x\cos x\,dx$$

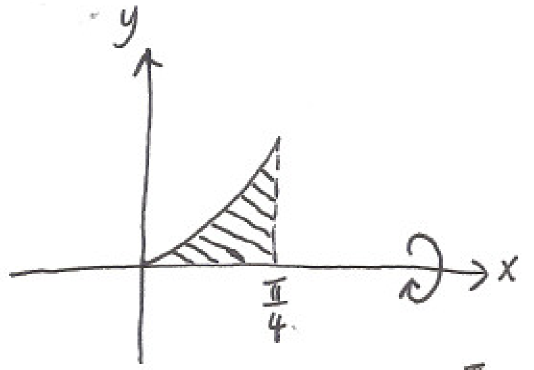
$$= \int_0^{\frac{\pi}{2}} x\cos x\,dx + \int_{\pi}^{\frac{\pi}{2}} x\cos x\,dx$$

$$= F(\tfrac{\pi}{2}) - F(0) + F(\tfrac{\pi}{2}) - F(\pi)$$

$$= \frac{\pi}{2} - 1 + \frac{\pi}{2} + 1$$

$$= \underline{\underline{\pi}}$$

8). E



$$\text{Vol.} = \int_0^{\frac{\pi}{4}} \pi\left(\sqrt{\tan x}\right)^2 dx = \pi \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= -\pi \ln \cos x \Big|_0^{\frac{\pi}{4}} = \pi\left\{-\ln \frac{1}{\sqrt{2}}\right\}$$

$$= \pi \ln \sqrt{2} = \underline{\underline{\frac{\pi}{2} \ln 2}}$$

9). E

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left\{\frac{1+(-1)^{n+1}+(-2)^n}{2^{n+1}}\right\} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{n!}\left\{\left(\frac{1}{2}\right)^n - \left(-\frac{1}{2}\right)^n + \left(\frac{-2}{2}\right)^n\right\}$$

$$= \frac{1}{2}\left\{\sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1/2)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\right\}$$

$$= \frac{1}{2}\left\{e^{1/2} - e^{-1/2} + e^{-1}\right\} \qquad \left(\because e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$$

$$= \underline{\underline{\frac{1}{2}\left(\sqrt{e} - \frac{1}{\sqrt{e}} + \frac{1}{e}\right)}}$$

10). C

$$\frac{1}{2-x} = \frac{1}{2-(x+3)+3} = \frac{1}{5-(x+3)} = \frac{1}{5}\left\{\frac{1}{1-\left(\frac{x+3}{5}\right)}\right\}$$

$$= \frac{1}{5}\sum_{n=0}^{\infty} \frac{(x+3)^n}{5^n} = \sum_{n=0}^{\infty} \frac{1}{5^{n+1}}(x+3)^n$$

$$\therefore a_5 = \frac{1}{5^{5+1}} = \frac{1}{5^6} = \underline{\underline{\frac{1}{15625}}}$$