Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2009-2010

MA1505 MATHEMATICS I

November 2009 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 1 (a) [5 marks]

Find the slope of the tangent to the curve

$$y = (35x - 69)^{43}$$

when $x = 2$.

Answer 1(a)	1505

(Show your working below and on the next page.)

$$y' = 43(35x - 69)^{42}(35)$$

$$x = 2 \Rightarrow y' = 43(70 - 69)^{42}(35) = \underline{\underline{1505}}$$

Question 1 (b) [5 marks]

Let

$$f(x) = ax^3 + bx^2$$

be a function defined on $(-\infty, \infty)$, where a and b are non-zero constants. Given that f has a point of inflection at $(1, 2)$, find the value of the product ab.

Answer 1(b)	-3

(Show your working below and on the next page.)

$f(x) = ax^3 + bx^2$

$f'(x) = 3ax^2 + 2bx$

$f''(x) = 6ax + 2b = 2(3ax + b)$

$f(1) = 2 \Rightarrow a + b = 2$

f'' changes sign at $x = 1 \Rightarrow 3a + b = 0$

$\therefore a = -1$

$b = 3$

$ab = \underline{\underline{-3}}$

Question 2 (a) [5 marks]

Let

$$f(x) = \frac{23 - 4x}{7 - 2x}$$

and let

$$\sum_{n=0}^{\infty} c_n (x-2)^n$$

be the Taylor series for f at $x = 2$. Find the **exact value** of $c_0 + c_{2009}$.

Answer 2(a)	$5 + \dfrac{2^{2009}}{3^{2008}}$

(Show your working below and on the next page.)

$$f(x) = \frac{14 - 4x + 9}{7 - 2x} = 2 + \frac{9}{7 - 2(x-2) - 4} = 2 + \frac{3}{1 - \frac{2}{3}(x-2)}$$

$$= 2 + 3\sum_{n=0}^{\infty} \frac{2^n}{3^n}(x-2)^n$$

$$= 5 + \sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}}(x-2)^n$$

$$c_0 + c_{2009} = 5 + \frac{2^{2009}}{3^{2008}}$$

Question 2 (b) [5 marks]

Use the method of separation of variables to find $u(x, y)$ that satisfies the partial differential equation

$$2u_{xy} = [\sin(x+y) + \sin(x-y)]\, u\ ,$$

given that $u(0,0) = 1$ and $u(\pi, \pi) = e^2$.

Answer 2(b)	$u = e^{1-\cos x+\sin y}$

(Show your working below and on the next page.)

Let $u = XY$, $X = X(x)$, $Y = Y(y)$.

$\therefore\ 2X'Y' = (2\sin x\cos y)XY$

$\therefore\ \dfrac{X'}{X\sin x} = \dfrac{Y\cos y}{Y'} = k$

$\dfrac{X'}{X} = k\sin x \Rightarrow \ln|X| = -k\cos x \Rightarrow X = Ae^{-k\cos x}$

$\dfrac{Y'}{Y} = \dfrac{1}{k}\cos y \Rightarrow \ln|Y| = \dfrac{1}{k}\sin y \Rightarrow Y = Be^{\frac{1}{k}\sin y}$

$\therefore\ u = Ce^{-k\cos x + \frac{1}{k}\sin y}$

$$\left.\begin{array}{l} u(0,0) = 1 \Rightarrow 1 = Ce^{-k} \\ u(\pi,\pi) = e^2 \Rightarrow e^2 = Ce^{k} \end{array}\right\} \Rightarrow k = 1,\ C = e$$

$\therefore\ \underline{\underline{u = e^{1-\cos x+\sin y}}}$

Question 3 (a) [5 marks]

Let

$$f(x) = x^2, \quad -\pi \leq x \leq \pi,$$

and $f(x + 2\pi) = f(x)$ for all x. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Find the **exact value** of $a_{2010} + b_{2010}$.

Answer 3(a)	$\frac{1}{(1005)^2}$

(Show your working below and on the next page.)

$$a_{2010} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos 2010x\, dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos 2010x\, dx$$

$$= \frac{1}{1005\pi}\int_0^{\pi} x^2 d(\sin 2010x) = -\frac{1}{1005\pi}\int_0^{\pi} 2x \sin 2010x\, dx$$

$$= \frac{1}{(1005)^2\pi}\int_0^{\pi} x\, d(\cos 2010x) = \frac{1}{(1005)^2}$$

f even $\Rightarrow b_{2010} = 0$

$$\therefore a_{2010} + b_{2010} = \frac{1}{(1005)^2}$$

Question 3 (b) [5 marks]

Let

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases}.$$

Find the **exact expression** of the first two non-zero terms in the sine Fourier half range expansion for $f(x)$.

Answer 3(b)	$\frac{6}{\pi} \sin \frac{\pi}{2} x - \frac{2}{\pi} \sin \pi x$

(Show your working below and on the next page.)

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$= \int_0^1 \sin \frac{n\pi x}{2} dx + \int_1^2 2 \sin \frac{n\pi x}{2} dx$$

$$= \left[-\frac{2}{n\pi} \cos \frac{n\pi x}{2} \right]_0^1 + \left[\frac{-4}{n\pi} \cos \frac{n\pi x}{2} \right]_1^2$$

$$= -\frac{2}{n\pi} \cos \frac{n\pi}{2} + \frac{2}{n\pi} - \frac{4}{n\pi} \cos n\pi + \frac{4}{n\pi} \cos \frac{n\pi}{2}$$

$$b_1 = \frac{2}{\pi} + \frac{4}{\pi} = \frac{6}{\pi}.$$

$$b_2 = \frac{1}{\pi} + \frac{1}{\pi} - \frac{2}{\pi} - \frac{2}{\pi} = -\frac{2}{\pi}$$

$$\therefore f(x) \sim \underline{\underline{\frac{6}{\pi} \sin \frac{\pi}{2} x - \frac{2}{\pi} \sin \pi x + \cdots}}$$

Question 4 (a) [5 marks]

Let S be the plane which passes through the points $(1,0,0)$, $(2,1,0)$ and $(3,2,1)$. Let L be the line which passes through $(0,0,0)$ and is parallel to the vector $-3\mathbf{i}+\mathbf{j}-\frac{26}{5}\mathbf{k}$. Find the coordinates of the point of intersection of L and S.

Answer 4(a)	$\left(\frac{3}{4}, -\frac{1}{4}, \frac{13}{10}\right)$

(Show your working below and on the next page.)

$\vec{u} = (2,1,0) - (1,0,0) = (1,1,0)$

$\vec{v} = (3,2,1) - (1,0,0) = (2,2,1)$

normal to $S = \vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{vmatrix} = \vec{i} - \vec{j}$

$S: \; x - y = 1$

$L: \; (x, y, z) = \left(-3t, t, -\frac{26}{5}t\right)$

$\therefore \; -3t - t = 1 \Rightarrow t = -\frac{1}{4}$

Ans. $\underline{\underline{\left(\frac{3}{4}, -\frac{1}{4}, \frac{13}{10}\right)}}$

Question 4 (b) [5 marks]

A space curve C is defined by the vector parametric equation

$$\mathbf{r}(t) = 2t^2\mathbf{i} + \left(t^2 - 4t\right)\mathbf{j} + (3t - 5)\,\mathbf{k}.$$

Let $\mathbf{T}$ denote the tangent vector to C at the point corresponding to $t = 1$. Find the length of the projection of $\mathbf{T}$ onto the vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Answer 4(b)	2

(Show your working below and on the next page.)

$$\vec{r}'(t) = 4t\,\vec{i} + (2t-4)\,\vec{j} + 3\vec{k}$$

$$\vec{T} = \vec{r}'(1) = 4\vec{i} - 2\vec{j} + 3\vec{k}$$

$$\left| \frac{\vec{T} \cdot (\vec{i} + 2\vec{j} + 2\vec{k})}{\sqrt{1+4+4}} \right| = \left| \frac{4-4+6}{3} \right| = \underline{\underline{2}}$$

Question 5 (a) [5 marks]

Let

$$f(x, y, z) = xy + yz + zx + 1505.$$

Find the **exact value** of the directional derivative of f at the point $(2, 3, 4)$ in the direction of the vector $\mathbf{u} = \mathbf{i} - \mathbf{j} - \sqrt{2}\mathbf{k}$.

Answer 5(a)	$\frac{1-5\sqrt{2}}{2}$

(Show your working below and on the next page.)

$$\nabla f = (y+z, \; x+z, \; y+x)$$

$$\nabla f(2,3,4) = (7, 6, 5)$$

$$D_{\vec{u}} f(2,3,4) = \nabla f(2,3,4) \cdot \vec{u} / \|\vec{u}\|$$

$$= \frac{(7,6,5) \cdot (1, -1, -\sqrt{2})}{\sqrt{1+1+2}}$$

$$= \frac{7-6-5\sqrt{2}}{2} = \underline{\underline{\frac{1-5\sqrt{2}}{2}}}$$

Question 5 (b) [5 marks]

Find the local maximum points, local minimum points, and saddle points, if any, of the function

$$f(x, y) = xy + (x + y)(120 - x - y).$$

Answer 5(b)	loc. max. (40, 40)

(Show your working below and on the next page.)

$$f = xy + 120x - x^2 - xy + 120y - xy - y^2$$
$$= 120x - x^2 - xy + 120y - y^2$$
$$f_x = 120 - 2x - y$$
$$f_y = 120 - x - 2y$$
$$f_x = f_y = 0 \Rightarrow x = y = 40$$

one critical point (40, 40).

$$f_{xx} = -2, \quad f_{xy} = -1, \quad f_{yy} = -2$$
$$f_{xx} f_{yy} - f_{xy}^2 = 4 - 1 = 3 = +ve$$

$\therefore$ (40, 40) is a local maximum point.

Question 6 (a) [5 marks]

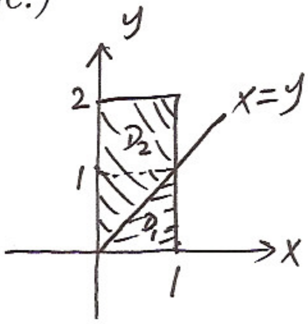
Find the **exact value** of the double integral

$$\int\int_D \sqrt{|x-y|}\,dxdy,$$

where D is the rectangular region: $0 \le x \le 1$ and $0 \le y \le 2$.

Answer 6(a)	$\frac{16}{15}\sqrt{2}$

(Show your working below and on the next page.)



On D_1, we have $x>y$

on D_2, we have $y>x$

$$\iint_D = \iint_{D_1} \sqrt{x-y}\,dxdy + \iint_{D_2} \sqrt{y-x}\,dxdy$$

$$= \int_0^1 \int_0^x \sqrt{x-y}\,dy\,dx + \int_0^1 \int_x^2 \sqrt{y-x}\,dy\,dx$$

$$= \int_0^1 \left[-\frac{2}{3}(x-y)^{3/2}\right]_{y=0}^{y=x} dx + \int_0^1 \left[\frac{2}{3}(y-x)^{3/2}\right]_{y=x}^{y=2} dx$$

$$= \int_0^1 \frac{2}{3}x^{3/2}\,dx + \int_0^1 \frac{2}{3}(2-x)^{3/2}\,dx$$

$$= \frac{2}{3}\left[\frac{2}{5}x^{5/2}\right]_0^1 - \frac{2}{3}\left[\frac{2}{5}(2-x)^{5/2}\right]_0^1$$

$$= \frac{4}{15} - \frac{4}{15} + \frac{4}{15}(2)^{5/2} = \underline{\underline{\frac{16}{15}\sqrt{2}}}$$

Question 6 (b) [5 marks]

Find the **exact value** of the iterated integral

$$\int_0^6 \left[\int_x^6 \frac{2xy}{\ln\{(1+y^2)^{(1+x^2)}\}} dy \right] dx \ .$$

Answer 6(b)	18

(Show your working below and on the next page.)

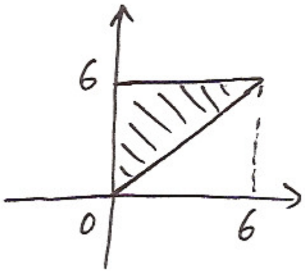
$$\int_0^6 \left[\int_x^6 \cdots dy \right] dx$$

$$= \int_0^6 \int_0^y \frac{2xy}{(1+x^2)\ln(1+y^2)} dx\, dy$$

$$= \int_0^6 \left[\ln(1+x^2) \right]_{x=0}^{x=y} \frac{y}{\ln(1+y^2)} dy$$

$$= \int_0^6 y\, dy$$

$$= \frac{1}{2}\left[y^2 \right]_0^6 = \underline{\underline{18}}$$



Question 7 (a) [5 marks]

Find the **exact value** of the volume of the solid enclosed laterally by the circular cylinder about z-axis of radius 1, bounded on top by the elliptic paraboloid

$$2x^2 + 4y^2 + z = 18\ ,$$

and bounded below by the plane $z = 0$.

Answer 7(a)	$\frac{33}{2}\pi$

(Show your working below and on the next page.)

$$\text{Vol} = \iint_{x^2+y^2\le 1} (18 - 2x^2 - 4y^2)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 (18 - 2r^2\cos^2\theta - 4r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_0^1 \int_0^{2\pi} \{18 - r^2(1+\cos 2\theta) - 2r^2(1-\cos 2\theta)\}\,d\theta\; r\,dr$$

$$= 2\pi \int_0^1 (18 - 3r^2)\, r\,dr$$

$$= 2\pi \left[9r^2 - \frac{3}{4}r^4\right]_0^1 = \underline{\underline{\frac{33}{2}\pi}}$$

Question 7 (b) [5 marks]

Find the **exact value** of the line integral

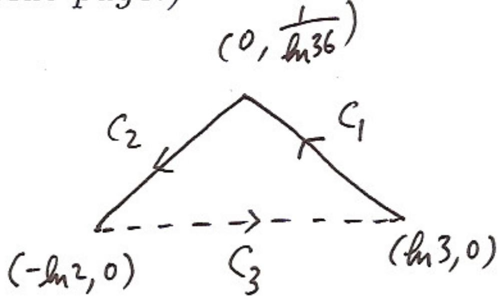
$$\int_C (e^x \cos y)\,dx + (2x - e^x \sin y)\,dy\ ,$$

where C consists of two line segments: C_1 from $(\ln 3, 0)$ to $\left(0, \frac{1}{\ln 36}\right)$, and C_2 from $\left(0, \frac{1}{\ln 36}\right)$ to $(-\ln 2, 0)$.

Answer 7(b)	-2

(Show your working below and on the next page.)

Apply Green's Theorem to the Δ:



$$\int_{C_1+C_2+C_3} (e^x \cos y)dx + (2x - e^x \sin y)\,dy$$

$$= \iint_\Delta \left[\frac{\partial}{\partial x}(2x - e^x \sin y) - \frac{\partial}{\partial y}(e^x \cos y)\right] dxdy$$

$$= \iint_\Delta 2\,dxdy = 2\text{ area}\,\Delta = 2 \times \frac{1}{2} \times (\ln 3 - (-\ln 2)) \times \frac{1}{\ln 36}$$

$$= 2 \times \frac{1}{2} \times \ln 6 \times \frac{1}{2\ln 6} = \frac{1}{2}$$

$$\therefore \int_{C_1+C_2} = \frac{1}{2} - \int_{C_3} = \frac{1}{2} - \int_{-\ln 2}^{\ln 3} e^t\,dt$$

$$= \frac{1}{2} - 3 + \frac{1}{2} = \underline{\underline{-2}}$$

Question 8 (a) [5 marks]

Find the **exact value** of the surface integral

$$\iint_S z\,dS,$$

where S is the surface $z = x^2 + y^2$ with $0 \le z \le 1$.

Answer 8(a)	$\left(\frac{5\sqrt{5}}{12} + \frac{1}{60}\right)\pi$

(Show your working below and on the next page.)

$S: \vec{r}(u,v) = u\vec{i} + v\vec{j} + (u^2+v^2)\vec{k}, \quad 0 \le u^2+v^2 \le 1.$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k}$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{4u^2+4v^2+1}$$

$$\iint_S z\,dS = \iint_{u^2+v^2\le 1} (u^2+v^2)\sqrt{4u^2+4v^2+1}\,du\,dv$$

$$= \int_0^{2\pi}\int_0^1 r^2\sqrt{4r^2+1}\,r\,dr\,d\theta$$

$$= 2\pi\int_1^{\sqrt{5}} \frac{t^2-1}{4}\, t\, \frac{1}{4}t\,dt \quad (\text{let } t = \sqrt{4r^2+1})$$

$$= \frac{\pi}{8}\int_1^{\sqrt{5}} (t^4 - t^2)\,dt$$

$$= \frac{\pi}{8}\left[\frac{1}{5}t^5 - \frac{1}{3}t^3\right]_1^{\sqrt{5}} = \frac{\pi}{8}\left\{5\sqrt{5} - \frac{5}{3}\sqrt{5} + \frac{2}{15}\right\}$$

$$= \frac{\pi}{8}\left(\frac{10\sqrt{5}}{3} + \frac{2}{15}\right) = \underline{\underline{\left(\frac{5\sqrt{5}}{12} + \frac{1}{60}\right)\pi}}$$

Question 8 (b) [5 marks]

Use Stokes' Theorem to find the **exact value** of the line integral

$$\oint_C (-yz\,dx + xz\,dy + xy\,dz)\,,$$

where C is the curve of intersection of the plane

$$x + y + z = 2$$

and the cylinder

$$x^2 + y^2 = 1,$$

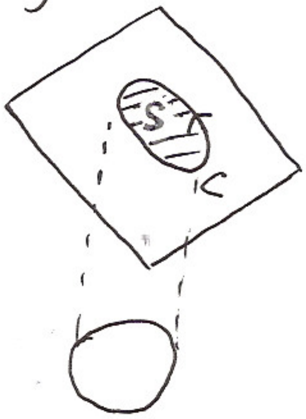
oriented in the counterclockwise sense when viewed from above.

Answer 8(b)	4π

(Show your working below and on the next page.)

Let S = part of the plane $\{x+y+z=2\}$ bounded by C.

S: $\vec{r}(u,v) = u\vec{i} + v\vec{j} + (2-u-v)\vec{k}$, $0 \le u^2+v^2 \le 1$.



$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}$$

curl $(-yz\vec{i} + xz\vec{j} + xy\vec{k})$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz & xz & xy \end{vmatrix} = -2y\vec{j} + 2z\vec{k}$$

(More working space for Question 8(b))

Observe $\vec{r}_u \times \vec{r}_v$ points upwards

$\therefore$ orientations of S and C are consistent

$$\oint_C = \iint_S \text{curl} \cdot \vec{dS} = \iint_{u^2+v^2\le 1} \{-2v + 2(2-u-v)\} \, du\, dv$$

$$= \iint_{u^2+v^2\le 1} (-4v + 4 - 2u) \, du\, dv$$

$$= \int_0^{2\pi} \int_0^1 (-4r\sin\theta + 4 - 2r\cos\theta) \, r\, dr\, d\theta$$

$$= 2\pi \int_0^1 4r\, dr$$

$$= 4\pi \left[r^2\right]_0^1 = \underline{\underline{4\pi}}$$

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