Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2008-2009

MA1505 MATHEMATICS I

November 2008 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 1 (a) [5 marks]

Find the slope of the tangent to the curve $x = t - \sin t$, $y = 1 - \cos t$, at the point corresponding to $t = \frac{\pi}{3}$.

Answer 1(a)	$\sqrt{3} \approx 1.732$

(Show your working below and on the next page.)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{\sin t}{1 - \cos t}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{\sqrt{3}/2}{1 - \frac{1}{2}}$$

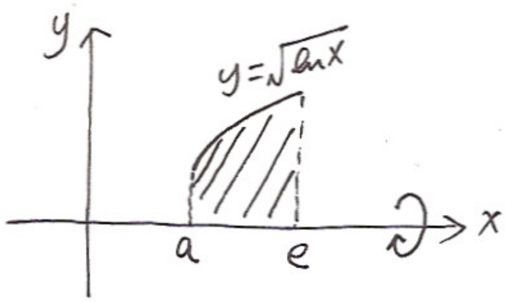
$$= \underline{\underline{\sqrt{3}}}$$

Question 1 (b) [5 marks]

Let a be a positive constant and $1 < a < e$. Let R denote the finite region in the first quadrant bounded by the curve $y = \sqrt{\ln x}$, the x-axis, the line $x = a$ and the line $x = e$. Find the **exact value** of the volume of the solid formed by revolving R one complete round about the x-axis. Leave your answer in terms of a.

Answer 1(b)	$\pi(a - a\ln a)$

(Show your working below and on the next page.)



$$\text{Vol} = \int_a^e \pi y^2\,dx$$

$$= \pi \int_a^e \ln x\,dx$$

$$= \pi \Big[x\ln x - x \Big]_a^e$$

$$= \underline{\underline{\pi(a - a\ln a)}}$$

Question 2 (a) [5 marks]

Let

$$f(x) = \frac{x^2+1}{x+1}$$

and let

$$\sum_{n=0}^{\infty} c_n (x+3)^n$$

be the Taylor series for f at $x = -3$. Find the **exact value** of $c_0 + c_1 + c_{101}$.

Answer 2(a)	$-\frac{9}{2} - \frac{1}{2^{101}}$

(Show your working below and on the next page.)

$$f(x) = \frac{x^2+1}{x+1}$$

$$= x - 1 + \frac{2}{x+1}$$

$$= (x+3) - 4 + \frac{2}{(x+3)-2}$$

$$= -4 + (x+3) - \frac{1}{1-\left(\frac{x+3}{2}\right)}$$

$$= -4 + (x+3) - \sum_{n=0}^{\infty} \frac{1}{2^n}(x+3)^n$$

$$= -5 + \frac{1}{2}(x+3) - \sum_{n=2}^{\infty} \frac{1}{2^n}(x+3)^n$$

$$\begin{array}{r} x-1 \\ x+1 \overline{)\, x^2 \quad +1} \\ \underline{x^2 + x} \\ -x+1 \\ \underline{-x-1} \\ +2 \end{array}$$

$$\therefore c_0 + c_1 + c_{101} = -5 + \frac{1}{2} - \frac{1}{2^{101}} = \underline{\underline{-\frac{9}{2} - \frac{1}{2^{101}}}}$$

Question 2 (b) [5 marks]

A car is moving with speed 20 m/s and acceleration α m/s^2 at a given instant. The car is observed to have moved a distance of 29 m in the next second. Using a second degree Taylor polynomial, estimate the value of α.

Answer 2(b)	18

(Show your working below and on the next page.)

We may assume that the car is at the origin with $t=0$ when $v = 20$ m/s and acceleration $= \alpha$ m/s^2.

Let x = distance from origin at time t.

$\therefore \frac{dx}{dt}(0) = 20, \quad \frac{d^2x}{dt^2}(0) = \alpha$

$\therefore x \simeq 0 + 20t + \frac{\alpha}{2!}t^2 = 20t + \frac{\alpha}{2}t^2$

$x = 29$ when $t = 1 \Rightarrow 29 = 20 + \frac{\alpha}{2}$

$\Rightarrow \alpha = \underline{\underline{18}}$

Question 3 (a) [5 marks]

Let

$$f(x) = x^2\sqrt{\pi^2 - x^2}, \quad -\pi \le x \le \pi,$$

and $f(x+2\pi) = f(x)$ for all x. Let

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Find the **exact value** of $b_2 + b_3 + \sum_{n=1}^{\infty} a_n$.

Answer 3(a)	$-\frac{\pi^4}{16}$

(Show your working below and on the next page.)

$\because f$ is even

$\therefore b_n = 0 \quad \forall n = 1, 2, 3, \cdots$

Put $x=0 \Rightarrow a_0 + \sum_{n=1}^{\infty} a_n = f(0) = 0$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x^2\sqrt{\pi^2 - x^2}\,dx$ (let $x = \pi\sin\theta$)

$= \frac{1}{\pi}\int_0^{\pi/2} (\pi^2\sin^2\theta)(\pi\cos\theta)(\pi\cos\theta\, d\theta)$

$= \frac{\pi^3}{4}\int_0^{\pi/2} \sin^2 2\theta\, d\theta$

$= \frac{\pi^3}{8}\int_0^{\pi/2} (1-\cos 4\theta)\, d\theta = \frac{\pi^4}{16}$

$\therefore b_2 + b_3 + \sum_{n=1}^{\infty} a_n = -a_0 = \frac{-\pi^4}{16}$

Question 3 (b) [5 marks]

Let $f(x) = x - 1$, $0 < x < 1$. Let

$$\sum_{n=1}^{\infty} b_n \sin n\pi x$$

be the sine Fourier half range expansion for $f(x)$. Find the **exact value** of b_{2008}.

Answer 3(b)	$-\frac{1}{1004\pi}$

(Show your working below and on the next page.)

$$b_{2008} = \frac{2}{1}\int_0^1 f(x) \sin 2008\pi x\, dx$$

$$= 2\left\{\int_0^1 x \sin 2008\pi x\, dx - \int_0^1 \sin 2008\pi x\, dx\right\}$$

$$= 2\left\{\frac{-1}{2008\pi}\int_0^1 x\, d(\cos 2008\pi x) + \frac{1}{2008\pi}\cos 2008\pi x\Big|_0^1\right\}$$

$$= -\frac{1}{1004\pi}\left\{x\cos 2008\pi x\Big|_0^1 - \int_0^1 \cos 2008\pi x\, dx\right\}$$

$$= -\frac{1}{1004\pi}\left\{1 - \frac{1}{2008\pi}\sin 2008\pi x\Big|_0^1\right\}$$

$$= -\frac{1}{1004\pi}$$

Question 4 (a) [5 marks]

Let S be the plane which passes through the points $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 3)$. Find the shortest distance from the point $(-1, -2, -3)$ to S.

Answer 4(a)	$\frac{24}{7}$

(Show your working below and on the next page.)

By inspection, or by a straight-forward calculation

$$S : \quad \frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$$

i.e. $6x + 3y + 2z = 6$

$$\therefore \text{distance} = \frac{|6(-1) + 3(-2) + 2(-3) - 6|}{\sqrt{6^2 + 3^2 + 2^2}}$$

$$= \frac{24}{7}$$

Question 4 (b) [5 marks]

Let **A** and **B** be two non-zero constant vectors and $||\mathbf{B}|| = 2$. If

$$\lim_{x\to\infty} (||x\mathbf{A} + \mathbf{B}|| - ||x\mathbf{A}||) = -\frac{1}{5},$$

find the **exact value** of $\cos\theta$, where θ is the angle between **A** and **B**.

Answer 4(b)	$-\frac{1}{10}$

(Show your working below and on the next page.)

$$\text{L.H.S.} = \lim_{x\to\infty} \frac{||xA+B||^2 - ||xA||^2}{||xA+B|| + ||xA||}$$

$$= \lim_{x\to\infty} \frac{(xA+B)\cdot(xA+B) - ||xA||^2}{||xA+B|| + ||xA||}$$

$$= \lim_{x\to\infty} \frac{2xA\cdot B + ||B||^2}{||xA+B|| + ||xA||}$$

$$= \lim_{x\to\infty} \frac{2A\cdot B + ||B||^2/x}{||A+\frac{B}{x}|| + ||A||} = \frac{A\cdot B}{||A||} = ||B|| \cos\theta$$

$$= 2\cos\theta$$

$$\therefore\ 2\cos\theta = -\frac{1}{5}$$

$$\cos\theta = -\frac{1}{10}$$

Question 5 (a) [5 marks]

Let $f(x, y, z)$ be a differentiable function of three variables, P be a point in space and $f(P) = 1$. It is known that the values of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ at P are equal to $-\sqrt{3}$, $-\frac{\sqrt{3}}{4}$, $-\frac{1}{\sqrt{12}}$ respectively. Suppose P moves 0.1 unit in the direction of the vector $\mathbf{u} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ to the point Q. Estimate the value of $f(Q)$.

Answer 5(a)	$\frac{113}{120} \approx 0.9417$

(Show your working below and on the next page.)

Let $\vec{v} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{3}}(\vec{i} - \vec{j} - \vec{k})$

$\nabla f(P) = -\sqrt{3}\,\vec{i} - \frac{\sqrt{3}}{4}\vec{j} - \frac{1}{\sqrt{12}}\vec{k}$

$\therefore D_{\vec{v}} f(P) = \nabla f(P) \cdot \vec{v} = -1 + \frac{1}{4} + \frac{1}{6} = -\frac{7}{12}$

$\therefore f(Q) - f(P) \approx (D_{\vec{v}} f(P))(0.1) = -\frac{7}{120}$

$f(Q) \approx f(P) - \frac{7}{120}$

$= 1 - \frac{7}{120}$

$= \frac{113}{120}$

Question 5 (b) [5 marks]

Let n be a fixed positive integer and $n \geq 2$. Find, if any, the local maximum points, the local minimum points and the saddle points of the function

$$f(x,y) = \ln(x^n y) - xy - (n-1)x,$$

where $x > 0$ and $y > 0$.

Answer 5(b)	$(1,1)$ = local max.

(Show your working below and on the next page.)

$$f_x = \frac{n x^{n-1} y}{x^n y} - y - (n-1) = \frac{n}{x} - y - (n-1) = 0 \quad \text{----} \; (1)$$

$$f_y = \frac{x^n}{x^n y} - x = \frac{1}{y} - x = 0 \Rightarrow y = \frac{1}{x} \quad \text{----} \; (2)$$

$$(1) \;\&\; (2) \Rightarrow \frac{n}{x} - \frac{1}{x} - (n-1) = 0 \Rightarrow x = 1 \quad (\because n \geq 2)$$

$\therefore (1,1)$ is the only critical point.

$$f_{xx} = -\frac{n}{x^2}, \quad f_{xy} = -1, \quad f_{yy} = -\frac{1}{y^2}$$

at $(1,1)$, $f_{xx} f_{yy} - (f_{xy})^2 = (-n)(-1) - 1 = n - 1 > 0 \quad (\because n \geq 2)$

$$f_{xx} = -n < 0$$

$\therefore (1,1)$ is a local maximum

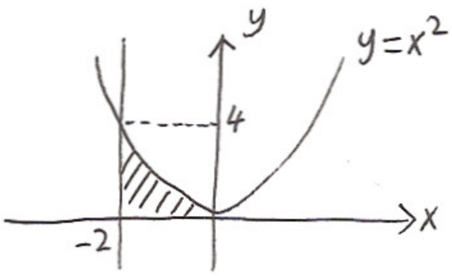
Question 6 (a) [5 marks]

Find the **exact value** of the integral

$$\int_0^4 \int_{-2}^{-\sqrt{y}} e^{x^3}\,dx\,dy.$$

Answer 6(a)	$\frac{1}{3} - \frac{1}{3}e^{-8}$

(Show your working below and on the next page.)



$$\int_0^4 \int_{-2}^{-\sqrt{y}} e^{x^3}\,dx\,dy = \int_{-2}^{0} \int_{0}^{x^2} e^{x^3}\,dy\,dx$$

$$= \int_{-2}^{0} x^2 e^{x^3}\,dx$$

$$= \frac{1}{3} e^{x^3} \Big|_{-2}^{0}$$

$$= \underline{\underline{\frac{1}{3} - \frac{1}{3} e^{-8}}}$$

Question 6 (b) [5 marks]

Find the **exact value** of the surface area of the part of the surface $z = 2 - x^2 - y^2$ which lies above the xy-plane.

Answer 6(b)	$\frac{13\pi}{3}$

(Show your working below and on the next page.)

$$z_x = -2x, \quad z_y = -2y$$

$$\text{Surface area} = \iint_{x^2+y^2 \le 2} \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy$$

$$= \iint_{x^2+y^2 \le 2} \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{1+4r^2}\, r\, dr\, d\theta$$

$$= 2\pi \left[\frac{1}{12} (1+4r^2)^{3/2} \right]_0^{\sqrt{2}}$$

$$= \frac{13\pi}{3}$$

Question 7 (a) [5 marks]

Let a be a positive odd integer. Evaluate the line integral

$$\int_C \mathbf{F} \bullet d\mathbf{r},$$

where $\mathbf{F} = \frac{y}{a}\mathbf{i} - \frac{x}{a}\mathbf{j} + \frac{2}{a}\mathbf{k}$ and C is the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from $(1, 0, 0)$ to $(-1, 0, a\pi)$.

Answer 7(a)	π

(Show your working below and on the next page.)

$$d\vec{r} = (-\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k})\,dt$$

$$\vec{F}(\vec{r}(t)) = \frac{\sin t}{a}\vec{i} - \frac{\cos t}{a}\vec{j} + \frac{2}{a}\vec{k}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{a\pi} \left(\frac{-\sin^2 t}{a} - \frac{\cos^2 t}{a} + \frac{2}{a} \right) dt$$

$$= \int_0^{a\pi} \frac{1}{a}\,dt$$

$$= \underline{\underline{\pi}}$$

Question 7 (b) [5 marks]

Find the **exact value** of the surface integral

$$\int\int_S (x+y)\,dS,$$

where S is the surface defined parametrically by

$$\mathbf{r}(u,v) = u\mathbf{i} + 3\cos v\mathbf{j} + 3\sin v\mathbf{k}, \qquad \left(0 \le u \le 4,\ 0 \le v \le \frac{\pi}{2}\right).$$

Answer 7(b)	$12\pi + 36$

(Show your working below and on the next page.)

$\vec{r}_u = \vec{i}$

$\vec{r}_v = -3\sin v\,\vec{j} + 3\cos v\,\vec{k}$

$\vec{r}_u \times \vec{r}_v = -3\sin v\,\vec{k} - 3\cos v\,\vec{j}$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{9\sin^2 v + 9\cos^2 v} = 3$

$\iint_S (x+y)\,dS = \int_0^{\pi/2}\int_0^4 (u + 3\cos v)(3)\,du\,dv$

$= 3\int_0^{\pi/2} (8 + 12\cos v)\,dv$

$= 3\left[8v + 12\sin v\right]_0^{\pi/2}$

$= \underline{\underline{12\pi + 36}}$

Question 8 (a) [5 marks]

Use Stokes' Theorem to find the **exact value** of the line integral

$$\oint_C \left(y\,dx - \frac{1}{2}z^2 dy + \frac{1}{2}x^2 dz \right),$$

where C is the curve of intersection of the plane $y + z = 0$ and the ellipsoid $3x^2 + 2y^2 + z^2 = 12$, oriented counterclockwise as seen from above.

Answer 8(a)	-4π

(Show your working below and on the next page.)

Let S be the region on $y+z=0$ and bounded by C.

$y+z=0$ and $3x^2+2y^2+z^2=12 \Rightarrow 3x^2+3y^2=12 \Rightarrow x^2+y^2=4$

$S: \vec{r}(u,v) = u\vec{i} + v\vec{j} - v\vec{k}, \quad u^2+v^2 \le 4.$

$\vec{r}_u = \vec{i}, \quad \vec{r}_v = \vec{j} - \vec{k}$

$\vec{r}_u \times \vec{r}_v = \vec{k} + \vec{j} = \vec{j} + \vec{k}$

$\therefore d\vec{S} = (\vec{j} + \vec{k})\,du\,dv$

$\because \vec{r}_u \times \vec{r}_v \cdot \vec{k} = 1 = +ve$

$\therefore \vec{r}_u \times \vec{r}_v$ points upwards

$\therefore$ the orientation of S agrees with the orientation of C.

(More working space for Question 8(a))

Let $\vec{F} = y\vec{i} - \frac{1}{2}z^2\vec{j} + \frac{1}{2}x^2\vec{k}$

$$\therefore \operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -\frac{1}{2}z^2 & \frac{1}{2}x^2 \end{vmatrix} = z\vec{i} - x\vec{j} - \vec{k}$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$$= \iint_{u^2+v^2\le 4} (-u-1)\,du\,dv$$

$$= \int_0^2 \int_0^{2\pi} (-r\cos\theta - 1)\,d\theta\,(r\,dr)$$

$$= \int_0^2 -2\pi r\,dr$$

$$= -\pi r^2 \Big|_0^2$$

$$= \underline{\underline{-4\pi}}$$

Question 8 (b) [5 marks]

Use the method of separation of variables to find $u(x,y)$ that satisfies the partial differential equation

$$u_x + u_y = (x-y)u,$$

given that $u(0,0) = u(0,2) = 1$.

Answer 8(b)	$u = e^{\frac{1}{2}x^2 - x - \frac{1}{2}y^2 + y}$

(Show your working below and on the next page.)

Let $u = X(x)Y(y)$

$\Rightarrow X'Y + XY' = (x-y)XY \Rightarrow \frac{X'}{X} + \frac{Y'}{Y} = x - y$

$\therefore \frac{X'}{X} - x = -\frac{Y'}{Y} - y = k$

$\therefore \frac{X'}{X} = x + k$ and $\frac{Y'}{Y} = -y - k$

$\ln|X| = \frac{1}{2}x^2 + kx + C_1$, $\ln|Y| = -\frac{1}{2}y^2 - ky + C_2$

$\therefore u = XY = Ce^{(\frac{1}{2}x^2 + kx - \frac{1}{2}y^2 - ky)}$

$u(0,0) = 1 \Rightarrow C = 1$

$u(0,2) = 1 \Rightarrow e^{-2-2k} = 1 \Rightarrow k = -1$

$\therefore u = e^{\frac{1}{2}x^2 - x - \frac{1}{2}y^2 + y}$