Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2007-2008

MA1505 MATHEMATICS I

November 2007 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 1 (a) [5 marks]

Find the slope of the tangent to the curve $y^2 = x^3 + 2x^2 - 20$ at the point $(3, 5)$.

Answer 1(a)	$\frac{39}{10}$

(Show your working below and on the next page.)

$$y^2 = x^3 + 2x^2 - 20$$

$$2yy' = 3x^2 + 4x$$

$$x = 3,\ y = 5 \Rightarrow 10y' = 27 + 12 = 39$$

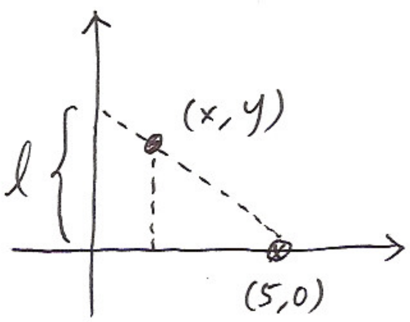
$$\therefore\ y' = \frac{39}{10}$$

Question 1 (b) [5 marks]

A lamp is located at the point $(5, 0)$ in the xy-plane. An ant is crawling in the first quadrant of the plane and the lamp casts its shadow onto the y-axis. How fast is the ant's shadow moving along the y-axis when the ant is at position $(1, 2)$ and moving so that its x-coordinate is increasing at a rate of $\frac{1}{2}$ units/sec and its y-coordinate is decreasing at a rate of $\frac{1}{5}$ units/sec?

Answer 1(b)	$\frac{1}{16}$

(Show your working below and on the next page.)



$$\frac{l}{y} = \frac{5}{5-x}$$

$$l = \frac{5y}{5-x}$$

$$\frac{dl}{dt} = \frac{5\frac{dy}{dt}(5-x) + 5y\frac{dx}{dt}}{(5-x)^2}$$

$$x=1,\ y=2,\ \frac{dx}{dt} = \frac{1}{2},\ \frac{dy}{dt} = -\frac{1}{5}$$

$$\Rightarrow \frac{dl}{dt} = \frac{5(-\frac{1}{5})(5-1) + 5(2)(\frac{1}{2})}{(5-1)^2} = \underline{\underline{\frac{1}{16}}}$$

Question 2 (a) [5 marks]

Find the **exact** value of the integral

$$\int_0^{\sqrt{101}} 2x^3 e^{x^2}\,dx.$$

Express your answer in terms of e.

Answer 2(a)	$100e^{101}+1$

(Show your working below and on the next page.)

Let $u = x^2$

$$\Rightarrow \int_0^{\sqrt{101}} 2x^3 e^{x^2}\,dx = \int_0^{101} u e^u\,du$$

$$= \int_0^{101} u\,d(e^u)$$

$$= \left[ue^u\right]_0^{101} - \int_0^{101} e^u\,du$$

$$= 101e^{101} - \left[e^u\right]_0^{101}$$

$$= \underline{\underline{100e^{101}+1}}$$

Question 2 (b) [5 marks]

Find a degree three polynomial to approximate the function

$$f(x) = \ln(1 + \sin x)$$

near $x = 0$.

Answer 2(b)	$x - \frac{1}{2}x^2 + \frac{1}{6}x^3$

(Show your working below and on the next page.)

$$f(x) = \ln(1+\sin x) \Rightarrow f(0) = 0$$

$$f'(x) = \frac{\cos x}{1+\sin x} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-\sin x(1+\sin x) - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-1}{1+\sin x} \Rightarrow f''(0) = -1$$

$$f'''(x) = \frac{\cos x}{(1+\sin x)^2} \Rightarrow f'''(0) = 1$$

$$\therefore f(x) \approx 0 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$$= \underline{\underline{x - \frac{1}{2}x^2 + \frac{1}{6}x^3}}$$

Question 3 (a) [5 marks]

Let $f(x) = |\sin x|$ for all $x \in (-\pi, \pi)$, and $f(x + 2\pi) = f(x)$ for all x. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Let m denote a fixed positive integer. Find the **exact** value of a_{2m}. Express your answer in terms of m in the simplest form.

Answer 3(a)	$-\dfrac{4}{(4m^2-1)\pi}$

(Show your working below and on the next page.)

Note that f is even.

$$a_{2m} = \frac{2}{\pi}\int_0^{\pi} \sin x \cos 2mx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \{\sin(2m+1)x - \sin(2m-1)x\} dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{2m+1}\cos(2m+1)x + \frac{1}{2m-1}\cos(2m-1)x \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{ \frac{(-1)^{2m+2}}{2m+1} + \frac{1}{2m+1} + \frac{(-1)^{2m-1}}{2m-1} - \frac{1}{2m-1} \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{2}{2m+1} - \frac{2}{2m-1} \right\}$$

$$= -\frac{4}{(4m^2-1)\pi}$$

Question 3 (b) [5 marks]

Find the shortest distance from the point $(-1, 1, 2)$ to the plane

$$2x + 3y - z - 10 = 0.$$

Answer 3(b)	$\frac{11}{\sqrt{14}}$

(Show your working below and on the next page.)

$$d = \frac{|2(-1) + 3(1) - (2) - 10|}{\sqrt{4+9+1}} = \frac{11}{\sqrt{14}}$$

Question 4 (a) [5 marks]

Let L_1 be a straight line which passes through the point $(-1, 0, 1)$ and suppose that L_1 is perpendicular to the plane $2x - y + 7z = 12$. Let L_2 be the line $\mathbf{r}(t) = (2+t)\,\mathbf{i} + (-4+2t)\,\mathbf{j} + (18-3t)\,\mathbf{k}$. Find the coordinates of the point of intersection of L_1 and L_2.

Answer 4(a)	$(3, -2, 15)$

(Show your working below and on the next page.)

L_1 : $(x, y, z) = (-1, 0, 1) + s(2, -1, 7)$

$= (-1+2s, -s, 1+7s)$

$$\begin{cases} 2+t = -1+2s & \text{---- ①} \\ -4+2t = -s & \text{---- ②} \\ 18-3t = 1+7s & \text{---- ③} \end{cases}$$

① + 2② $\Rightarrow$ $-6+5t = -1$

$\Rightarrow t = 1$

$\therefore$ point of intersection $=$ $\underline{\underline{(3, -2, 15)}}$

Question 4 (b) [5 marks]

Let $f(x,y) = \ln(\tan x + \tan y)$, with $0 < x, y < \frac{\pi}{2}$. Find the value of

$$(\sin 2x)\frac{\partial f}{\partial x} + (\sin 2y)\frac{\partial f}{\partial y}.$$

Your answer should be a number.

Answer 4(b)	2

(Show your working below and on the next page.)

$$\frac{\partial f}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y}$$

$$\frac{\partial f}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y}$$

$$\sin 2x \frac{\partial f}{\partial x} + \sin 2y \frac{\partial f}{\partial y} = \frac{2\tan x}{\tan x + \tan y} + \frac{2\tan y}{\tan x + \tan y}$$

$$= \underline{\underline{2}}$$

Question 5 (a) [5 marks]

Let n be a positive integer. Find the directional derivative of

$$f(x, y) = x^2 - xy + y^n$$

at the point $(2, 1)$ in the direction of the vector joining the point $(2, 1)$ to the point $(6, 4)$. Express your answer in terms of n.

Answer 5(a)	$\frac{3n+6}{5}$

(Show your working below and on the next page.)

$$\vec{u} = \frac{(6,4)-(2,1)}{\|(6,4)-(2,1)\|} = \frac{(4,3)}{5} = \left(\frac{4}{5}, \frac{3}{5}\right)$$

$$\nabla f = (f_x, f_y) = (2x - y, -x + ny^{n-1})$$

$$\nabla f(2,1) = (3, n-2)$$

$$D_{\vec{u}} f(2,1) = \nabla f(2,1) \cdot \vec{u}$$

$$= \frac{12}{5} + \frac{3(n-2)}{5}$$

$$= \frac{3n+6}{5}$$

Question 5 (b) [5 marks]

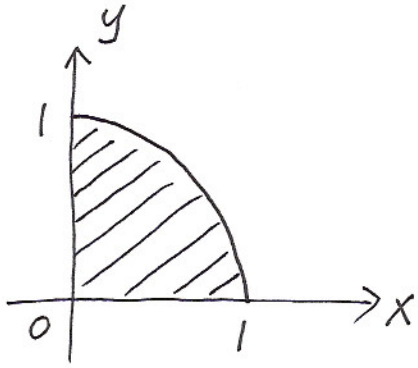
Evaluate

$$\iint_D x\,dA,$$

where D is the finite plane region in the first quadrant bounded by the two coordinate axes and the curve $y = 1 - x^2$.

Answer 5(b)	$\frac{1}{4}$

(Show your working below and on the next page.)



$$\iint_D x\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x^2} x\,dy\,dx$$

$$= \int_0^1 \left[x y\right]_{y=0}^{y=1-x^2} dx$$

$$= \int_0^1 x(1-x^2)\,dx$$

$$= \int_0^1 (x - x^3)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \underline{\underline{\frac{1}{4}}}$$

Question 6 (a) [5 marks]

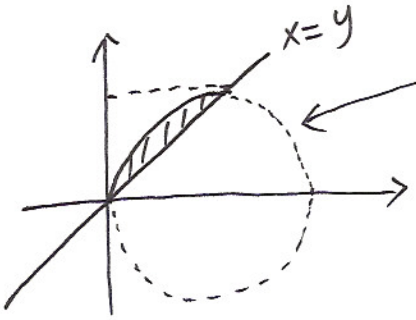
Find the **exact** value of the integral

$$\int_0^1 \int_{1-\sqrt{1-y^2}}^{y} y e^{\left(x^2-\frac{2}{3}x^3\right)} dx dy.$$

Express your answer in terms of e.

Answer 6(a)	$\frac{1}{2}(e^{1/3}-1)$

(Show your working below and on the next page.)



$$x = 1-\sqrt{1-y^2} \iff \sqrt{1-y^2} = 1-x$$
$$\iff 1-y^2 = (1-x)^2$$
$$= 1-2x+x^2$$
$$\iff y^2 = 2x - x^2$$

$$\text{Given integral} = \int_0^1 \int_x^{\sqrt{2x-x^2}} y e^{(x^2-\frac{2}{3}x^3)} dy\, dx$$

$$= \int_0^1 \left[\frac{1}{2} y^2 e^{(x^2-\frac{2}{3}x^3)}\right]_{y=x}^{y=\sqrt{2x-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 (2x-2x^2)\, e^{(x^2-\frac{2}{3}x^3)} dx$$

$$= \frac{1}{2} e^{(x^2-\frac{2}{3}x^3)} \Big|_0^1$$

$$= \underline{\underline{\frac{1}{2}(e^{1/3}-1)}}$$

Question 6 (b) [5 marks]

Let a be a positive constant. Evaluate the line integral

$$\int_C (x^2 + y^2 + z^2)\, ds,$$

where C is the circular helix given by $x = a\cos t$, $y = a\sin t$, $z = t$, $0 \le t \le a$.

Answer 6(b)	$\frac{4}{3}a^3\sqrt{1+a^2}$

(Show your working below and on the next page.)

$C: \vec{r}(t) = (a\cos t,\ a\sin t,\ t)$

$\vec{r}\,'(t) = (-a\sin t,\ a\cos t,\ 1)$

$\|\vec{r}\,'(t)\| = \sqrt{1+a^2}$

$$\int_C (x^2+y^2+z^2)\,ds = \int_0^a (a^2+t^2)\sqrt{1+a^2}\,dt$$

$$= \sqrt{1+a^2}\left[a^2 t + \frac{1}{3}t^3\right]_0^a$$

$$= \underline{\underline{\frac{4}{3}a^3\sqrt{1+a^2}}}$$

Question 7 (a) [5 marks]

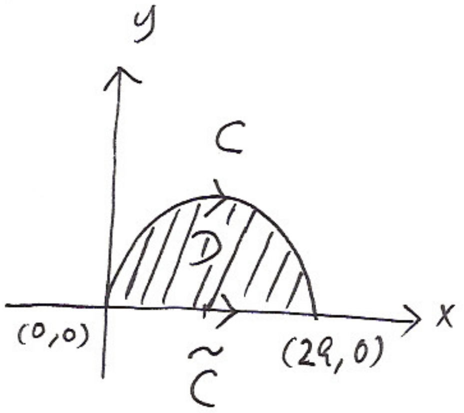
Let a be a positive constant. Evaluate the line integral

$$\int_C (2xe^{\sin y} + 3x^2y^2 + ay)\,dx + (x^2e^{\sin y}\cos y + 2x^3y + 2ax + 1)\,dy,$$

where C is the semicircle, centered at $(a, 0)$ with radius a, in the first quadrant joining $(0, 0)$ to $(2a, 0)$.

Answer 7(a)	$4a^2 - \frac{1}{2}\pi a^3$

(Show your working below and on the next page.)



Let $\tilde{C}$: $\vec{r}(t) = (t, 0)$, $0 \le t \le 2a$.

Then $\partial D = \tilde{C} - C$

Apply Green's Theorem to D:

Let $P = 2xe^{\sin y} + 3x^2y^2 + ay$

$Q = x^2e^{\sin y}\cos y + 2x^3y + 2ax + 1$

$$\oint_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (2xe^{\sin y}\cos y + 6x^2y + 2a) - (2xe^{\sin y}\cos y + 6x^2y + a)\,dA$$

$$= \iint_D a\,dA = a(\text{area } D) = \frac{1}{2}\pi a^3$$

$\therefore \int_{\tilde{C}} - \int_C Pdx + Qdy = \frac{1}{2}\pi a^3$

$\therefore \int_C Pdx + Qdy = \left\{\int_{\tilde{C}} Pdx + Qdy\right\} - \frac{1}{2}\pi a^3 = \left\{\int_0^{2a} 2t\,dt\right\} - \frac{1}{2}\pi a^3$

$$= \underline{\underline{4a^2 - \frac{1}{2}\pi a^3}}$$

Question 7 (b) [5 marks]

Evaluate the surface integral

$$\iint_S \mathbf{F} \bullet d\mathbf{S},$$

where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the portion of the paraboloid $z = 1 - x^2 - y^2$ lying on and above the xy plane. The orientation of S is given by the outer normal vector.

Answer 7(b)	$\dfrac{3\pi}{2}$

(Show your working below and on the next page.)

S : $\vec{r}(u,v) = (u, v, 1-u^2-v^2)$

$\vec{r}_u = (1, 0, -2u)$, $\vec{r}_v = (0, 1, -2v)$

$\vec{r}_u \times \vec{r}_v = 2u\vec{i} + 2v\vec{j} + \vec{k}$.

at $(0,0,1)$, $\vec{r}_u \times \vec{r}_v = \vec{k}$ points outwards.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{u^2+v^2 \le 1} \{2u^2 + 2v^2 + (1-u^2-v^2)\}\, du\, dv$$

$$= \int_0^{2\pi}\int_0^1 (1+r^2)\, r\, dr\, d\theta$$

$$= 2\pi \left[\tfrac{1}{2}r^2 + \tfrac{1}{4}r^4\right]_0^1$$

$$= \underline{\underline{\frac{3\pi}{2}}}$$

Question 8 (a) [5 marks]

By using Stokes' Theorem, or otherwise, find the **exact** value of the surface integral

$$\int\int_S (\nabla \times \mathbf{F}) \bullet d\mathbf{S},$$

where S is the hemisphere $x^2 + y^2 + z^2 = 16$ lying on and above the xy plane, and $\mathbf{F} = \left(x^2 + y - 4e^z\right)\mathbf{i} + \left(3xy\cos^2 z\right)\mathbf{j} + \left(2e^{xy}\sin z + x^2yz^3\right)\mathbf{k}$. The orientation of S is given by the outer normal vector. Express your answer in terms of π.

Answer 8(a)	-16π

(Show your working below and on the next page.)

Let C: $\vec{r}(t) = 4\cos t\,\vec{i} + 4\sin t\,\vec{j} + 0\vec{k}$, $0 \le t \le 2\pi$.

Note that the orientation of C is anti-clockwise and this matches with the outer normal orientation of S.

By Stokes' Theorem

$$\iint_S (\nabla\times\vec{F})\cdot dS = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} \{(16\cos^2 t + 4\sin t - 4)(-4\sin t) + 48\sin t\cos t\,(4\cos t)\}\,dt$$

$$= \int_0^{2\pi} (128\cos^2 t\sin t - 16\sin^2 t + 16\sin t)\,dt$$

$$= \left[-\frac{128}{3}\cos^3 t\right]_0^{2\pi} - 8\int_0^{2\pi}(1-\cos 2t)\,dt$$

$$= \underline{\underline{-16\pi}}$$

Question 8 (b) [5 marks]

Find a solution of the form $u(x,y) = F(ax+y)$, where a is a constant and F is a differentiable single variable function, to the partial differential equation

$$u_x - 2u_y = 0,$$

that satisfies the condition $u(x,0) = \cos x$.

Answer 8(b)	$u(x,y) = \cos \frac{2x+y}{2}$

(Show your working below and on the next page.)

$$u_x = aF'(ax+y)$$

$$u_y = F'(ax+y)$$

$$u_x - 2u_y = 0 \Rightarrow aF'(ax+y) - 2F'(ax+y) = 0$$

$$\Rightarrow a = 2$$

$$\therefore u(x,y) = F(2x+y)$$

$$u(x,0) = \cos x \Rightarrow F(2x) = \cos x$$

$$\Rightarrow F(x) = \cos \frac{x}{2}$$

$$\therefore u(x,y) = F(2x+y) = \cos \frac{2x+y}{2}$$

$$\therefore u(x,y) = \cos \frac{2x+y}{2}$$