NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2012-2013

MA1505 Mathematics 1

May 2013 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **SIX (6)** printed pages.
2. **This is a CLOSED BOOK examination. One A4-sized helpsheet is allowed.**
3. Answer **ALL** 8 questions. The marks for each question are indicated at the beginning of the question. The maximum score is **80 marks**.
4. **Write your matriculation number neatly on the front page of the answer booklet provided.**
5. **Write your solutions in the answer booklet. Begin your solution to each question on a new page.**
6. Calculators may be used. However, you should lay out systematically the various steps in your calculations.

Question 1 [10 marks]

(a) Given

$$f(x,y,z) \;=\; 2(xz)^{1015} \;+\; 23\sin\left(\frac{3}{10}\,y\right),$$

find the values of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ when $x = 1,\;\; y = 0$ and $z = 1$.

(b) The weekly revenue R of a production firm is modelled by a function

$$R(K,L) \;=\; 75\,K^{1/3}\,L^{2/3}\,,$$

where K is capital expenditure and L is labour cost. The quantities R, K and L are in *thousands** of dollars. Suppose that capital expenditure is \$125,000 and labour cost is \$8,000. If capital expenditure is increasing at \$750 per week and revenue is to increase at \$13,000 per week, find the required rate of increase of labour cost.

(**Caution: capital expenditure of \$125,000 means that* $K = 125$.)

Question 2 [10 marks]

(a) Suppose a metal plate is placed on the xy-plane with the temperature T at the point $P(x,y)$ on the plate given by

$$T(x,y) \;=\; \frac{x^2+1250}{y^2+16}\,.$$

At the point $P(25,-3)$, the direction in which the temperature increases most rapidly is given by

$$\mathbf{i} \;+\; \beta\mathbf{j}.$$

Find the exact value of β.

(b) Let C be the curve in the xy-plane given by

$$C \; : \; x^2 + xy + 2y^2 \; = \; 56.$$

Consider a point $P(x_0, y_0, z_0)$ on the plane Π given by

$$\Pi \; : \; x + 3y - z \; = \; 0.$$

If P lies above C, find the largest possible value of z_0.

Question 3 [10 marks]

(a) Let

$$f(x) \; = \; \begin{cases} 1 \text{ if } & -\dfrac{\pi}{4} < x < \dfrac{\pi}{4} \\[2ex] 0 \text{ if } & -\pi < x < -\dfrac{\pi}{4} \quad \text{or} \quad \dfrac{\pi}{4} < x < \pi, \end{cases}$$

with $f(x + 2\pi) = f(x)$ for all x.

Suppose that the Fourier series of $f(x)$ is

$$f(x) \; = \; a_0 \; + \; \sum_{n=1}^{\infty} a_n \cos nx.$$

Find the exact value of $\displaystyle\sum_{n=1}^{1505} n\, a_n$.

(b) Let $f(x) \; = \; e^{-x}$ for $0 < x < \pi$.

Suppose that the sine half range expansion of $f(x)$ is

$$f_s(x) \; = \; \sum_{n=1}^{\infty} b_n \sin nx.$$

Find the exact value of b_n, giving the answer in terms of n.

Question 4 [10 marks]

(a) Find the exact value of the iterated integral

$$\int_0^{1505} \int_0^{2} x^{1505} e^{x^{1506}} \, dx \, dy.$$

(b) The (finite) region R in the first quadrant is bounded by the graphs of

$$y = 12x \quad \text{and} \quad y = x^2.$$

If S is the upper cone given by

$$S \ : \ z \ = \ \sqrt{x^2 + y^2}\,,$$

find the exact value of the surface area of the part of S that lies over R.

Question 5 [10 marks]

(a) The region R in the first quadrant is bounded by the following lines and curve:

$$x = 0, \quad y = 2 \quad \text{and} \quad y = \sqrt[3]{x}.$$

Find the exact value of the double integral

$$\iint_R \frac{x}{\sqrt{y^7 + 16}} \, dA.$$

(b) The solid region D is bounded above by the elliptic paraboloid

$$S_1 \ : \ z \ = \ 24 - 5x^2 - 6y^2$$

and below by the parabolic cylinder

$$S_2 \ : \ z \ = \ x^2.$$

Find the exact volume of D.

Question 6 [10 marks]

(a) A curved vertical fence is placed on the xy-plane such that the base of the fence is described parametrically by the curve

$$C \ : \ x = 3t, \quad y = t^4, \quad \text{where} \ \ 0 \leq t \leq 1.$$

The height of the fence at the point (x, y) is given by

$$h(x, y) \ = \ 24xy.$$

Find the total surface area (front surface and back surface) of the fence.

(*All lengths are in metres. Ignore the thickness of the fence.*)

(b) Let C be a circle in the xy-plane. If

$$\mathbf{F}(x, y) \ = \ \left(x + x^2 y - 4y\right) \mathbf{i} \ + \ \left(1 - xy^2\right) \mathbf{j}\,,$$

find the largest possible value of the line integral $\displaystyle\oint_C \mathbf{F} \bullet d\mathbf{r}$.

Question 7 [10 marks]

(a) Let S be the upper hemisphere given by

$$S \ : \ x^2 + y^2 + z^2 \ = \ 4, \quad z \geq 0.$$

Find the exact value of the surface integral $\displaystyle\iint_S z^3 \, dS$.

(b) Let S be the portion of the circular paraboloid

$$S \ : \ z \ = \ x^2 + y^2, \quad \text{where} \ \ z \leq 16.$$

If

$$\mathbf{F}(x, y, z) \ = \ x\mathbf{i} \ + \ y\mathbf{j} \ + \ z\mathbf{k},$$

find the exact value of the surface integral $\displaystyle\iint_S \mathbf{F} \bullet d\mathbf{S}$, where the orientation of S is the inward normal vector.

Question 8 [10 marks]

(a) Let the curve C be the intersection of the plane Π with the upper cone S, where

$$\Pi \; : \; z \; = \; x+2 \qquad \text{and} \qquad S \; : \; z \; = \; \sqrt{2x^2+y^2}\,.$$

Viewed from above, C is oriented in the anticlockwise direction. If

$$\mathbf{F}(x,y,z) \; = \; 7z\mathbf{i} \; + \; 6x\mathbf{j} \; - \; 5y\mathbf{k},$$

use Stokes' Theorem to find the exact value of the line integral $\oint_C \mathbf{F} \bullet d\mathbf{r}$.

(b) Let S be the circular cylinder given by

$$S \; : \; x^2+y^2 \; = \; 1, \quad \text{where} \;\; 0 \le z \le 1.$$

(*Note that the cylinder is open at both ends.*)

If

$$\mathbf{F}(x,y,z) \; = \; z^4 e^y \mathbf{i} \; + \; \left(19y - 4x^2 y + \sin x^3\right)\mathbf{j} \; + \; 4x^2 z\mathbf{k},$$

find the exact value of the surface integral $\iint_S \mathbf{F} \bullet d\mathbf{S}$, where the orientation of S is given by the outward normal vector.

END OF PAPER