NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2011-2012

MA1505 Mathematics 1

April 2012 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **SIX (6)** printed pages.
2. **This is a CLOSED BOOK examination. One A4-sized helpsheet is allowed.**
3. Answer **ALL** 8 questions. The marks for each question are indicated at the beginning of the question. The maximum score is **80 marks**.
4. **Write your matriculation number neatly on the front page of the answer booklet provided.**
5. **Write your solutions in the answer booklet. Begin your solution to each question on a new page.**
6. Calculators may be used. However, you should lay out systematically the various steps in your calculations.

Question 1 [10 marks]

(a) Given

$$f(x) \;=\; x^{2012}\sin 1505x \quad \text{for} \;\; x > 0,$$

find the value of $f'(\pi)$.

(b) The region R in the first quadrant is bounded by the curve $y = \sqrt{x}$, the line $y = x - 2$ and the x-axis. If R is revolved about the line $x = 4$, find the exact volume of the solid generated.

Question 2 [10 marks]

(a) Using suitable Taylor series, or otherwise, find the exact value of

$$\sum_{n=0}^{\infty} \frac{1+8^n}{n!}\,.$$

(b) Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{4^n + (-2)^n}{4n-2}\, x^{4n-2}\,.$$

Question 3 [10 marks]

(a) Let $f(x) = x^2$ for $-2 < x < 2$ and $f(x+4) = f(x)$ for all x.

The Fourier series of $f(x)$ is

$$\frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{2}.$$

(*This Fourier series need not be derived.*)

Use the above Fourier series to find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

Hence find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(*Give the exact values in terms of* π.)

(b) Let $f(x) = \sin x$ for $-1 < x < 1$ and $f(x+2) = f(x)$ for all x.

Find the Fourier series of $f(x)$.

Question 4 [10 marks]

(a) The plane Π and line L have respective equations:

$$\begin{aligned} \Pi &: x + y + z = 15, \\ L &: \mathbf{r}(t) = 6\mathbf{i} + \mathbf{j} + 3\mathbf{k} + t(\mathbf{i} - \mathbf{j} - \mathbf{k}), \end{aligned}$$

where t is any real number. Find the point of intersection of Π and L.

(b) The curve C in the xy-plane is the portion of the graph of

$$y = e^x, \quad \text{where } \frac{1}{2}\ln 3 \le x \le \frac{1}{2}\ln 8.$$

Find the exact length of C.

Question 5 [10 marks]

(a) Suppose a metal plate is placed on the xy-plane such that the temperature at the point (x, y) is

$$T(x,y) \;=\; 100 - 16x^2 - 2y^2.$$

A heat-seeking particle P, initially placed at $(1,1)$, moves in the direction of maximum temperature increase at each point. The path of P is a curve with equation $y = f(x)$. If $f(x)$ is a differentiable function such that

$$\frac{dy}{dx} = k\left(\frac{y}{x}\right),$$

where k is a constant, find the value of k.

(*The function $f(x)$ need not be found.*)

(b) Find the local maximum, local minimum and saddle points, if any, of

$$f(x,y) \;=\; x^3 - 3x^2 + y - e^y.$$

Question 6 [10 marks]

(a) Find the exact value of the iterated integral

$$\int_0^6 \int_{y/3}^2 x\cos\left(x^3\right)\,dx\,dy.$$

(b) The solid region D is bounded above by the sphere

$$S_1 \;:\; x^2 + y^2 + z^2 \;=\; 2$$

and below by the paraboloid

$$S_2 \;:\; z \;=\; x^2 + y^2.$$

Find the exact volume of D.

Question 7 [10 marks]

(a) The closed curve C in the xy-plane consists of the upper semicircle

$$x^2 + y^2 = 16, \quad \text{where } y \geq 0,$$

and the line segment joining $(-4, 0)$ to $(4, 0)$. A force field

$$\mathbf{F}(x, y) = \left(2xy + \cos\left(x^2\right) + x^8\right)\mathbf{i} + \left(x^2 + 3xy + e^y\right)\mathbf{j},$$

moves a particle, which traverses C once in the anticlockwise direction. Use Green's Theorem to find the work done as a line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

(b) The curve C in the xy-plane has vector equation

$$\mathbf{r}(t) = (\cos 2\pi t)\,\mathbf{i} + (\ln t)\,\mathbf{j}, \quad \text{where } 1 \leq t \leq 2.$$

If

$$\mathbf{F}(x, y) = (e^y + ye^x)\,\mathbf{i} + (xe^y + e^x)\,\mathbf{j},$$

find the exact value of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Question 8 [10 marks]

(a) Let S be the portion of the cone

$$z = \sqrt{x^2 + y^2}, \quad \text{where } 1 \leq z \leq 2.$$

Find the exact value of the surface integral $\iint_S z^2\, dS$.

(b) The curve C is the intersection of the plane

$$\Pi \ : \ 4y - z = 0$$

and the paraboloid

$$S \ : \ z = x^2 + y^2.$$

Viewed from above, C is oriented in the anticlockwise direction. If

$$\mathbf{F}(x, y, z) \ = \ x^3\mathbf{i} \ + \ 2x\mathbf{j} \ + \ z^2\mathbf{k},$$

find the exact value of the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

END OF PAPER