NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2010-2011

MA1505 Mathematics 1

April/May 2011 — Time allowed : 2 hours

Matriculation Number:

INSTRUCTIONS TO CANDIDATES

1. **Write your matriculation number neatly in the space above.**
2. Do not insert loose papers into this booklet. This booklet will be collected at the end of the examination.
3. This examination paper contains a total of **EIGHT (8)** questions and comprises **THIRTY-THREE (33)** printed pages.
4. Answer **ALL** 8 questions. The marks for each question are indicated at the beginning of the question. The maximum score is **80 marks**.
5. **Write your solution in the space below each question.**
6. Calculators may be used. However, you should lay out systematically the various steps in your calculations.
7. **This is a CLOSED BOOK examination. One A4-sized helpsheet is allowed.**

For official use only. Do not write in the boxes below.

Question	1	2	3	4	5	6	7	8
Marks								

Question 1 (a) [5 marks]

The curve $\mathcal{C}$ in the xy-plane is given parametrically by the equations

$$\mathcal{C} \quad : \quad x = e^{2t} + 2 \ , \quad y = e^{2t} - 8t, \qquad \text{where } t \text{ is any real number.}$$

Find all local extreme points $P(x, y)$, if any, on $\mathcal{C}$, and determine whether these points are local maximum or local minimum.

(*More space for the solution to Question 1 (a).*)

Question 1 (b) [5 marks]

The region R lies in the first quadrant and is bounded by the y-axis and the curves $y = e^{2x}$ and $y = 2e^{x}$. If R is revolved about the line $y = 4$, find the volume of the solid generated.
(*Give the exact volume in terms of* π.)

(*More space for the solution to Question 1 (b).*)

Question 2 (a) [5 marks]

Suppose k is a positive constant such that the power series

$$1 \ + \ k(x-13) \ + \ k^2(x-13)^2 \ + \ \cdots \ + \ k^n(x-13)^n \ + \ \cdots$$

only converges for all values of x with $8 < x < 18$.
Find the value of k.

(*More space for the solution to Question 2 (a).*)

Question 2 (b) [5 marks]

Let $f(x)$ be a function such that $f(0) = 1505$ and

$$f'(x) = \ln\left[\frac{2+x^2}{2-x^2}\right].$$

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is the Taylor series of $f(x)$ at $x = 0$, find the exact value of c_{2011}.

(*More space for the solution to Question 2 (b).*)

Question 3 (a) [5 marks]

Let $f(x) = \begin{cases} 1+x & \text{if} \quad -1 < x < 0 \\ 0 & \text{if} \quad 0 < x < 1 \end{cases}, \qquad f(x+2) = f(x)$ for all x.

Suppose that the Fourier series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x).$$

Find the exact value of $a_0 + a_1$.

(*More space for the solution to Question 3 (a).*)

Question 3 (b) [5 marks]

Let $f(x) = e^x$ for $0 < x < 1$.

Suppose that the Fourier sine series of $f(x)$ is

$$f_s(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

Find the exact value of b_1.

(More space for the solution to Question 3 (b).)

Question 4 (a) [5 marks]

The point P lies on the line L with parametric equations

$$L \quad : \quad x = 5t, \quad y = 3t, \quad z = t, \qquad \text{where } t \text{ is any real number.}$$

Given that the distance from P to the plane Π with Cartesian equation

$$\Pi \quad : \quad 3x + 4y + 12z \;=\; 0$$

is **54** and P lies above the xy-plane, find the coordinates of P.

(*More space for the solution to Question 4 (a).*)

Question 4 (b) [5 marks]

Let $f(x,y)$ be a function with continuous partial derivatives. At the origin, suppose that the largest directional derivative of $f(x,y)$ is $D_{\mathbf{u}}f(0,0)$, where

$$\mathbf{u} = \frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}.$$

If $D_{\mathbf{u}}f(0,0)$ is nonzero, find all possible directions, if any, expressed as a unit vector $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j}$, such that

$$D_{\mathbf{w}}f(0,0) = \frac{1}{2}\,D_{\mathbf{u}}f(0,0).$$

(*More space for the solution to Question 4 (b).*)

Question 5 (a) [5 marks]

In a triangle OXY, the side OY of length 40 cm is kept constant while the side OX of length x cm and the angle $\theta = \angle XOY$ vary over time.

At a particular moment, $x = 20$ and is increasing at 4 cm/sec, while $\theta = \dfrac{\pi}{6}$ radian and is decreasing at $\dfrac{1}{10}$ radian/sec.

Find the exact instantaneous rate of change of the area of triangle OXY.

(*More space for the solution to Question 5 (a).*)

Question 5 (b) [5 marks]

Find the local maximum, local minimum and saddle points, if any, of

$$f(x,y) \; = \; (x-2)\ln(xy), \quad \text{where } x>0, \quad y>0.$$

(*More space for the solution to Question 5 (b).*)

Question 6 (a) [5 marks]

Find the exact value of the iterated integral

$$\int_0^4 \int_{\sqrt{y}}^2 x^2\sqrt{4+x^5}\, dxdy.$$

(*More space for the solution to Question 6 (a).*)

Question 6 (b) [5 marks]

Let R be a region in the xy-plane. Find the largest possible exact value of the integral

$$\iint_R (4 - x^2 - y^2)\, dA.$$

(*More space for the solution to Question 6 (b).*)

Question 7 (a) [5 marks]

An object M at the point $P(x, y, z)$ is moved by a force $\mathbf{F}$ given by

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + 2\mathbf{j} + 4z\mathbf{k}.$$

Find the work done $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$ when M is moved from the point $A(1, 0, 0)$ to the point $B(1, 0, 2\pi)$ along the circular helix $\mathcal{C}$ given by

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$

(*Give the exact value in terms of* π.)

(*More space for the solution to Question 7 (a).*)

Question 7 (b) [5 marks]

Let $\mathcal{C}$ be the portion of the ellipse $x^2 + 4y^2 = 4$ that joins $A(0, 1)$ to $B(2, 0)$ in the first quadrant. If

$$\mathbf{F}(x, y) = y^2\mathbf{i} + \left(2xy - e^{2y}\right)\mathbf{j},$$

find the exact value of the line integral $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$, giving the answer in terms of e.

(*More space for the solution to Question 7 (b).*)

Question 8 (a) [5 marks]

Let S be the cone described by

$$z = \sqrt{x^2 + y^2}, \quad \text{where} \ \ 0 \leq z \leq 8.$$

If

$$\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + xyz\mathbf{k},$$

use Stokes' Theorem to find the surface integral $\iint_S \text{curl}(\mathbf{F}) \bullet d\mathbf{S}$, where the orientation of S is given by the outer normal vector.

(*More space for the solution to Question 8 (a).*)

Question 8 (b) [5 marks]

Let S be the upper hemisphere with equation

$$S \quad : \quad z \ = \ \sqrt{9 - x^2 - y^2}.$$

If

$$\mathbf{F}(x, y, z) \ = \ 2xye^z\mathbf{i} \ - \ y^2e^z\mathbf{j} \ + \ (2 + 5z)\mathbf{k},$$

find the surface integral $\displaystyle\iint_S \mathbf{F} \bullet d\mathbf{S}$, where the orientation of S is given by the upward normal vector.

(*More space for the solution to Question 8 (b).*)

END OF PAPER