NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2009-2010

MA1505 Mathematics 1

April/May 2010 — Time allowed : 2 hours

Matriculation Number:

INSTRUCTIONS TO CANDIDATES

1. **Write your matriculation number neatly in the space above.**
2. Do not insert loose papers into this booklet. This booklet will be collected at the end of the examination.
3. This examination paper contains a total of **EIGHT (8)** questions and comprises **THIRTY-THREE (33)** printed pages.
4. Answer **ALL** 8 questions. The marks for each question are indicated at the beginning of the question. The maximum score is **80 marks**.
5. **Write your solution in the space below each question.**
6. Calculators may be used. However, you should lay out systematically the various steps in your calculations.
7. **This is a CLOSED BOOK examination. One A4-sized helpsheet is allowed.**

For official use only. Do not write in the boxes below.

Question	1	2	3	4	5	6	7	8
Marks								

Question 1 (a) [5 marks]

The curve $\mathcal{C}$ is given by the equation

$$\mathcal{C} \quad : \quad y \;=\; (\ln x)^5 \;\;, \quad \text{where } x > 0.$$

The line L is tangent to $\mathcal{C}$ at $x = e^2$. If the point $(0, t)$ lies on L, find the value of t.

(More space for the solution to Question 1 (a).)

Question 1 (b) [5 marks]

The region R is bounded by the parabola $x = y^2 + 1$ and the line $x = 5$. Find the volume of the solid generated when R is revolved about the y-axis. (*Give the exact volume in terms of* π.)

(*More space for the solution to Question 1 (b).*)

Question 2 (a) [5 marks]

Given the power series

$$1 + \frac{1}{8}(x-3) + \frac{1}{64}(x-3)^2 + \cdots + \left(\frac{x-3}{8}\right)^n + \cdots$$

find all the values of x for which the series converges, giving the answer in the form $a < x < b$.

(*More space for the solution to Question 2 (a).*)

Question 2 (b) [5 marks]

By integrating a Taylor series of $\dfrac{x}{e^x}$, find the exact value of the sum

$$\sum_{n=0}^{\infty}(-1)^n \frac{2^{3n}}{n!\,(n+2)} \; .$$

(*Give the exact value in terms of* e.)

(*More space for the solution to Question 2 (b).*)

Question 3 (a) [5 marks]

Let $f(x) = x(1-x)$ for $0 < x < 1$.

Suppose that the sine half range expansion of $f(x)$ is

$$f_s(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

Find the value of b_5.

(*More space for the solution to Question 3 (a).*)

Question 3 (b) [5 marks]

Let $g(x) = x^4$ for $-1 < x < 1$ and $g(x+2) = g(x)$ for all x.

The Fourier series of $g(x)$ is

$$g(x) \;=\; \frac{1}{5} \;+\; \frac{8}{\pi^4}\sum_{n=1}^{\infty}(-1)^n\,\frac{n^2\pi^2-6}{n^4}\,\cos n\pi x\,.$$

Given the sum of the series $S = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} \;=\; \frac{\pi^2}{12}$, use this Fourier series to find the sum of the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}.$$

(*The above series S and the Fourier series need not be derived. Give the exact value of the sum in terms of π.*)

(*More space for the solution to Question 3* (*b*).)

Question 4 (a) [5 marks]

Let Π be the plane with Cartesian equation

$$\Pi \quad : \quad 2x + y + 2z = 0.$$

Find Cartesian equations of the two planes that are parallel to Π and are of (shortest) distance **6** from Π.

(*More space for the solution to Question 4 (a).*)

Question 4 (b) [5 marks]

The curve $\mathcal{C}$ in the xy-plane has equation

$$\mathcal{C} \quad : \quad 9y^2 \ = \ 4x^3$$

with end-points $A(0,0)$ and $B(9,18)$. Find the exact length of $\mathcal{C}$.

(*More space for the solution to Question 4 (b).*)

Question 5 (a) [5 marks]

On a certain mountain, the elevation z above a point (x, y) in an xy-plane at sea level is

$$z = f(x, y) = 3205 - 0.02x^2 - 0.01y^2,$$

where x, y, and z are in metres. The positive x-axis points east, and the positive y-axis points north. A mountain climber is at the point $P(200, 300, 1505)$.

Find the direction, given as a unit vector $a\mathbf{i} + b\mathbf{j}$, of the steepest ascent.

(*More space for the solution to Question 5 (a).*)

Question 5 (b) [5 marks]

Let p and q be constants such that $pq > 0$ and let

$$f(x, y) = xy + \frac{p^3}{x} + \frac{q^3}{y}, \qquad \text{where } x \neq 0, \quad y \neq 0.$$

Find the local maximum, local minimum and saddle points (if any) of $f(x, y)$.
(*Give the answers in terms of* p *and* q.)

(*More space for the solution to Question 5 (b).*)

Question 6 (a) [5 marks]

Find the exact value of the iterated integral

$$\int_0^1 \int_{e^y}^{e} \frac{x}{\ln x}\, dx dy.$$

(*More space for the solution to Question 6 (a).*)

Question 6 (b) [5 marks]

The sphere S is given by

$$S \quad : \quad x^2 \; + \; y^2 \; + \; (z+3)^2 \; = \; 25.$$

The solid region D lies above the xy-plane and is bounded by S and the xy-plane. Find the exact volume of D in terms of π.

(*More space for the solution to Question 6* (*b*).)

Question 7 (a) [5 marks]

The force $\mathbf{F}$ exerted by a certain electric charge placed at the origin on a charged particle E at a point $P(x, y, z)$ is

$$\mathbf{F}(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

The straight line segment $\mathcal{C}$ joins the points $A(1, 0, 0)$ and $B(1, 2, 2)$.

Find the work done $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$ when E is moved from A to B along $\mathcal{C}$.

(*More space for the solution to Question 7 (a).*)

Question 7 (b) [5 marks]

Let S be the triangle with vertices $P(2,0,0)$, $Q(0,2,0)$, and $R(0,0,2)$. The boundary $\mathcal{C}$ of S has an anticlockwise orientation when viewed from above S. Let

$$\mathbf{F}(x,y,z) = \left(x - 6y^2\right)\mathbf{i} + \left(y - 6z^2\right)\mathbf{j} + \left(z - 6x^2\right)\mathbf{k}.$$

Use Stokes' Theorem to find line integral $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$.

(*More space for the solution to Question 7 (b).*)

Question 8 (a) [5 marks]

Let S be the sphere with equation

$$S \quad : \quad x^2 + y^2 + z^2 \;=\; 1.$$

If

$$\mathbf{F}(x, y, z) \;=\; (6x + e^y)\,\mathbf{i} \;+\; (7y + e^z)\,\mathbf{j} \;+\; (8z + e^x)\,\mathbf{k},$$

use the Divergence Theorem to find the surface integral $\iint_S \mathbf{F} \bullet d\mathbf{S}$, where the orientation of S is given by the outer normal vector.

(*More space for the solution to Question 8 (a).*)

Question 8 (b) [5 marks]

Use the method of separation of variables to find $u(x, y)$ that satisfies the partial differential equation

$$u_x - u_y \;=\; (x^3 - y^3)u,$$

given that $u(0,0) = e^9$ and $u(2,2) = e^{18}$.

(*More space for the solution to Question 8 (b).*)

END OF PAPER