NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2008-2009

MA1505 Mathematics 1

April/May 2009 — Time allowed : 2 hours

Matriculation Number:

INSTRUCTIONS TO CANDIDATES

1. **Write your matriculation number neatly in the space above.**
2. Do not insert loose papers into this booklet. This booklet will be collected at the end of the examination.
3. This examination paper contains a total of **EIGHT (8)** questions and comprises **THIRTY-THREE (33)** printed pages.
4. Answer **ALL** 8 questions. The marks for each question are indicated at the beginning of the question. The maximum score is **80 marks**.
5. **Write your solution in the space below each question.**
6. Calculators may be used. However, you should lay out systematically the various steps in your calculations.
7. **This is a CLOSED BOOK examination. One A4-sized helpsheet is allowed.**

For official use only. Do not write in the boxes below.

Question	1	2	3	4	5	6	7	8
Marks								

Question 1 (a) [5 marks]

The tangent to the curve

$$x^3 + y^3 = 9xy$$

at the point $(4, 2)$ meets the line $y = x$ at the point P. Find the coordinates of P.

(*More space for the solution to Question 1 (a).*)

Question 1 (b) [5 marks]

Let k be a positive constant. The region R is bounded by the lines

$$x = k, \qquad x = k\sqrt{3}, \qquad y = 0,$$

and the curve

$$y = \frac{1}{\sqrt{x^2 + k^2}}\,.$$

Find the volume of the solid generated when R is revolved about the x-axis. (*Give the exact value in terms of k and π.*)

(*More space for the solution to Question 1 (b).*)

Question 2 (a) [5 marks]

For $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, the series

$$\sin^2 x \ + \ \sin^4 x \ + \ \sin^6 x \ + \ \cdots \ + \ \sin^{2k} x \ + \ \cdots$$

converges. Find its sum.

(*More space for the solution to Question 2 (a).*)

Question 2 (b) [5 marks]

Let

$$g(x) \;=\; \frac{x-16}{x^2-16x+68}\,.$$

If $f(x)$ is a function such that $f(8) = 8$ and $f\,'(x) = g(x)$, then use Taylor series to find the value of $f^{(2009)}(8)$.
(*You may give your answer in terms of factorials.*)

(*More space for the solution to Question 2 (b).*)

Question 3 (a) [5 marks]

Let $f(x) = \begin{cases} 0 & \text{if } -3 < x \leq 0 \\ x & \text{if } \quad 0 < x < 3 \end{cases}, \quad f(x+6) = f(x)$ for all x.

Suppose that the Fourier series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{3} + b_n \sin \frac{n\pi x}{3} \right).$$

Find the positive value of n such that

$$2 + 27\pi^2 a_n = 0.$$

(*More space for the solution to Question 3 (a).*)

Question 3 (b) [5 marks]

Let $u(t) = \begin{cases} 0 & \text{if } -1 < t \le 0 \\ \sin \pi t & \text{if } \quad 0 < t < 1 \end{cases}$, $\quad u(t+2) = u(t)$ for all t.

The Fourier series of $u(t)$ is

$$u(t) = \frac{1}{\pi} + \frac{1}{2}\sin \pi t - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} \cos 2n\pi t\,.$$

Use this Fourier series to find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)}$.

Hence, find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$. (*You need not derive the above Fourier series. Give the exact value of each sum in terms of* π.)

(*More space for the solution to Question 3 (b).*)

Question 4 (a) [5 marks]

The space curves C_1 and C_2 are described as follows:

$C_1: \quad \mathbf{r}_1(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k},$

$C_2: \quad \mathbf{r}_2(u) = (\sin u)\mathbf{i} + (\cos u)\mathbf{j} + 2u\mathbf{k},$

where t and u are real numbers. Find all the points of intersection between C_1 and C_2 that lie between the two planes $z = 0$ and $z = 2\pi$.

(*More space for the solution to Question 4 (a).*)

Question 4 (b) [5 marks]

The two planes Π_1 and Π_2 are given by:

$$\Pi_1 : \quad x + y - 2z = 0, \qquad\qquad \Pi_2 : \quad 3x - y + 4z = 6.$$

The line L is the intersection of Π_1 and Π_2. If the plane Π passes through the point $P(8, 0, 0)$ and is perpendicular to L, find a Cartesian equation of Π.

(*More space for the solution to Question 4 (b).*)

Question 5 (a) [5 marks]

A circular cylinder is changing in such a way that its radius r is increasing at the rate of 1 cm/min and its height h is decreasing at the rate of 2 cm/min. When $r = 7$ cm and $h = 5$ cm, find the rate at which the volume V of the cylinder is changing.
(*Give the exact rate in terms of* π.)

(*More space for the solution to Question 5 (a).*)

Question 5 (b) [5 marks]

Find the local maximum, local minimum and saddle points (if any) of

$$f(x, y) \; = \; x^3 - 3xy - y^3.$$

(*More space for the solution to Question 5 (b).*)

Question 6 (a) [5 marks]

Find the value of the iterated integral

$$\int_0^9 \int_{\sqrt{y}}^3 e^{x^3}\, dx dy.$$

(*More space for the solution to Question 6 (a).*)

Question 6 (b) [5 marks]

The region D lies inside the upper hemisphere $z = \sqrt{25 - x^2 - y^2}$, but outside the cylinder $x^2 + y^2 = 16$. Find the exact volume of D in terms of π.

(*More space for the solution to Question 6 (b).*)

Question 7 (a) [5 marks]

The curve C is the boundary of the region in the first quadrant bounded by the curves

$$x = 0, \qquad y = 8, \qquad y = x^3.$$

If

$$\mathbf{F}(x, y) \;=\; (e^x + xy)\,\mathbf{i} \;+\; \left(3x^2 + \sin^4 y\right)\mathbf{j},$$

find line integral $\oint_C \mathbf{F} \bullet d\mathbf{r}$, where C is oriented in the anticlockwise direction.

(*More space for the solution to Question 7 (a).*)

Question 7 (b) [5 marks]

Let S be the portion of the plane $x + 4y + z = 12$ that lies in the first octant ($x \geq 0,\ y \geq 0,\ z \geq 0$). Let C be the boundary of S with anticlockwise orientation when viewed from above S. If

$$\mathbf{F}(x, y, z) \ = \ (z - x)\,\mathbf{i} \ + \ (x - y)\,\mathbf{j} \ + \ (y - z)\,\mathbf{k},$$

use Stokes' Theorem to find line integral $\oint_C \mathbf{F} \bullet d\mathbf{r}$.

(*More space for the solution to Question 7 (b).*)

Question 8 (a) [5 marks]

The base of a circular fence is described parametrically by the curve

$$C: \quad x = 8\cos t, \quad y = 8\sin t, \quad \text{where } \ 0 \leq t \leq 2\pi.$$

The height of the fence at the point (x, y) is given by

$$h(x, y) \ = \ 4 \ + \ \frac{1}{32}xy.$$

Find the total surface area (front surface and back surface) of the fence.
(*All lengths are in metres. Ignore the thickness of the fence. Give the exact surface area in terms of* π.)

(*More space for the solution to Question 8 (a).*)

Question 8 (b) [5 marks]

Use the method of separation of variables to find $u(x, y)$ that satisfies the partial differential equation

$$u_{xy} = (xy - y - x + 1)u, \quad \text{where} \ x > 1, \ y > 1,$$

given that $u(2, 2) = 2$ and $u(4, 4) = 2e^8$.

(*More space for the solution to Question 8* (*b*).)

END OF PAPER